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ABSTRACT

The ten papers in this document were developed for a meeting prepared for the Third International Congress on Mathematical Education. Each paper is concerned with research from 1970-1975 related to the mathematical learning process. The first paper describes projects conducted in England on both content and process learning. The second paper discusses the concrete/semi-concrete/abstract movement, cognitive-affective emphases, and goal structure for learning mathematics. The third paper includes a general description of recent developments in research on individual students, teacher effects, and process-outcome studies. A fourth paper focuses on probabilistic thinking in children and adolescents. The fifth paper looks at the relation between the individual, the environment, and the school as affecting learning. The sixth paper identifies organizations and journals in the United States which are concerned with research in mathematics education. The seventh paper categorizes research on mathematics learning into five general groups: research internal to the field of mathematics education, research by academic and educational psychologists, clinical research, artificial intelligence and computer simulation, and structural learning. An eighth paper summarizes the major psychological influences on the problems of mathematics curriculum and instruction and explores the impact of information-processing theory on psychology and education. The ninth paper (written in German) covers Piagetian theory, cognitive psychology, microprocesses in mathematical learning, and motivation. The final paper discusses the setting for research in the United States and summarizes research on organization and sequencing of instruction, teaching approaches, developmental stages, diagnosis and remediation, algorithmic learning, problem solving, use of materials, achievement evaluation, and attitudes. (DT)

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Forschung zum Prozeß des Mathematiklernens

Reihe: Materialien und Studien
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"As educational researchers, we find ourselves in the mildly embarrassing position of knowing less than we have proven."

Gene V. Glass,

presidential address to the
1976 Annual Meeting of the
American Educational Research
Association, San Francisco

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ZU DIESEM HEFT

Der in Karlsruhe stattfindende 3. Internationale Kongreß für Mathematikdidaktik (3rd Int. Congress on Mathematics Education) sieht umfangreiche Trend-Reports zu zentralen Themen der Dikdaktik der Mathematik vor. Dieses Teilprogramm wird unter Einarbeitung der Diskussionsergebnisse vom Kongreß später von der UNESCO in Vol. IV "Trends in Mathematics Education" veröffentlicht. Einer dieser insgesamt dreizehn Überblicksberichte soll der "Forschung zum Prozeß des Mathematiklernens" gewidmet werden. Der Bericht soll

- die Entwicklungen und Trends der letzten vier Jahre darstellen,
- eine kritische Analyse auf der Grundlage einer theoretischen Orientierung geben und
- Vorschläge für zukünftige Forschung und Entwicklungsarbeiten machen.

Am Institut für Didaktik der Mathematik der Universität Bielefeld fand im Mai 1976 eine von der Arbeitsgruppe F1 (siehe den Strukturplan des IDM) geplante internationale Arbeitstagung zu diesem Thema statt. Zur Vorbereitung dieser Tagung und zugleich als Vorarbeit für den Kongreßbericht wurde von den eingeladenen Teilnehmern ein Katalog von Fragen in offener Form bearbeitet. Hierbei interessierten insbesondere drei Punkte:

(1) Die wichtigsten Untersuchungen zum Prozeß des Mathematiklernens im Zeitraum etwa von 1970 bis 1975, ihre Ergebnisse und die Gründe für ihre Bedeutung.

(2) Wichtige Institutionen und Projekte, die Forschung zum Mathematiklernen betreiben.

(3) Mögliche Defizite in der Entwicklung und Empfehlungen für die künftige Forschung.

Zur Dokumentation der Arbeitstagung werden die eingegangenen Beiträge in diesem Materialienband veröffentlicht. Die Literaturhinweise und Referenzen aller Beiträge wurden zusammengefaßt und in diesen Band aufgenommen.

Heinrich Bauersfeld

RESEARCH RELATED TO THE MATHEMATICAL LEARNING PROCESS

A. W. Bell

The mathematical learning process comprises

a) content learning, and

b) process learning.

These may be subdivided into:

A1 the learning of concepts and relationships

A2 the learning of facts and skills

B1 the learning of general mathematical strategies (e.g. of problem-solving, generalisation, proof)

B2 the development of critical and creative powers in a mathematical context

B3 the development of (i) a general appreciation of the nature of mathematics and of its applications in the world, and of (ii) positive attitudes towards it.

Under each heading, research can help (1) to describe the learning outcomes and to show how to evaluate them, and (2) to identify the most effective teaching methods for achieving them.

Most past research has concentrated on comparing teaching methods without sufficient differentiation between the different types of outcome desired. Some of the most significant recent research (in my view) has attempted more satisfactory descriptions of the aspects of learning. Recent work at Nottingham relates to A1 and B1; under A1, a project on the Acquisition of Structural Concepts has shown that the description of mathematical content with regard to number needs to specify not mere-

ly mathematical concepts such as counting numbers, rational numbers etc., with the four usual operations and applications of these to practical situations, but to pay equal regard to numero-spatial structures such as numbered and lettered lines, sequences of labelled points, clock-type structures and so on. In the course of this project several questions arose concerning the relation between symbols and their referent concepts. In general it appeared that difficulties which showed themselves as an inability to manipulate symbols correctly, on examination seemed to stem from limited experience of the symbolic language, so that changes which went almost unnoticed by the teacher, such as from A $\square$ = B to B = A $\square$, produced symbolic statements which the pupil could no longer interpret in the referent situation. (See Reference 1).

Another project, conducted by Dr. K.F.Collis at Nottingham during 1974, on the Development of Formal Reasoning, showed that not only in the field of numbers and their applications but also in algebraic manipulation the psychological dimensions of difficulty did not correspond with an adult view of the processes. The child's logic led to matching and substituting processes which did not correspond to the processes normally taught. Also, while 10-12 year old children regarded a literal expression as verified by the substitution of one small number, or perhaps two, the 14 year olds had a sense that it had to be valid for all of a loosely understood general set. (Reference 5).

Under the heading B1, the learning of general mathematical strategies, Mr. A.W.Bell has been studying the relationship between formal proof and pupils' informal explanations of simple generalisations discovered or tested by them. Stages have been identified, ranging from empirical inference to deductive explanation, some particular strategies have been provisionally

shown to be relatively easily learnable, and others more resistant to improvement, and some of the conditions for improved performance exposed. (Reference 2). A small scale curriculum development project with an emphasis on the development of strategies of investigation and generalisation has also been mounted, and some pilot evaluations. of this are being conducted. (Reference 3).

A start towards bringing together the insights from these and other researches to form a coherent theory of mathematical learning has been made by Professor E.A. Lunzer. (Reference 4).

REFERENCES

1 Shell Centre for Mathematical Education : Composition of Integers and Modular Arithmetic. University of Nottingham, 1975

2 Bell A.W. : A Study of Pupils' Proof-Explanations in Mathematical Situations. interim paper, 1976

3 Bell A.W. : General Strategies in the South Nottinghamshire Project. interim paper, 1976

3 Bell A.W. : Evaluation of Attainments of General Mathematical Strategies. interim paper, 1976

4 Lunzer E.A. : Towards an Epistomological Theory of Mathematics Learning. discussion paper, March 1976

5 Collis K.F. : The Development of Formal Reasoning. University of Newcastle, Australia, 1975

RESEARCH RELATED TO THE MATHEMATICAL LEARNING PROCESS

Desmond Broomes

1. INTRODUCTION:

During the early 1970's certain trends in research related to the mathematical learning process seemed to have emerged; and this emergence may be associated with two major factors. First, a search for a sensitive language to grasp the value issues that permeate curriculum development has led to a more useful definition of curriculum in terms and activities that should enable schools to become directly in tune with their communities. Hence the nature of curriculum activities in many countries is operationalized by attempts to achieve a cultural synthesis (Maclure, 1972).

Second, new insights for interpreting research findings obtained in culturally different settings are being acquired by researchers as they work on the frontiers of educational reforms and are being used to guide and inform successful undertakings. For example, certain categories of teacher behaviour seem inappropriate for certain cultures and do not appear to have fruitful relationships with pupils' cognitive development. (Kleinfeld, 1975).

2. IMPORTANT RESEARCH TRENDS OR CLUSTERS OF ACTIVITIES (1970-1975)

Three important clusters of research activities derived from certain philosophical and cultural positions, may be identified in the published literature and 'grey papers' of the period under review. These research trends or clusters are:

2.1 The Concrete/Semi-concrete/Abstract Movement

This trend reflects the gradual understanding by teachers (curriculum developers and researchers) that it is the growth of the child's general ways of knowing that determines the manner in which new knowledge in mathematics is assimilated, and hence what opportunities should be provided for organising, maintaining, and strengthening the strategies of thinking in mathematics that are available within the child's cognitive structure at any moment.

(1) The existence of three distinct modes of representation and combinations of these modes. These modes are commonly referred to as concrete (or manipulative or enactive), semi-concrete (or pictorial or ikonic), and abstract (or symbolic).

(2) This importance of learning activities following a scheme that emphasizes the effectiveness of the role of each mode and the interplay of two or three modes in learning mathematics at the primary level (Armstrong, 1972; Beilin, 1971; Boersig, 1973; Carmody, 1971; Curry, 1971;, Fennema, 1972).

BEILIN (1971) made the point that a mathematical idea which requires a level of thought beyond that which the child possesses, can only be learned partially or with much difficulty, and at best the child's grip on the concept will be tenuous.

ARMSTRONG (1972) investigated the general hypothesis that

> when a concrete (or enactive or manipulative) mode is used as the instructional intervention, pupils who are at the beginning stages of representational thought should exhibit greater learning on mathematical concepts which require representational thought.

Her subjects were trainable mentally retarded persons of mental age 2-4 years and educable mentally retarded persons of mental age 5:8 - 11:9 years. The findings

showed that mode of intervention (manipulative vs non-manipulative) had its greatest impact on the child in the very beginning stages of representational thought (when the mathematics he was learning specifically required representational thought).

FENNEMA (1972) investigated the relative effectiveness of concrete (manipulative or enactive) and symbolic modes of presentation. Selected 7-8 year olds were taught a previously unlearned mathematical principle. At the end of the instructional period all groups of children had learned the principle to the point of direct recall. Groups of children who had learned with the symbolic model were able to transfer their learning better than those groups which had learned with the concrete model. Her tentative conclusions were that for certain groups of children learning a specific principle, a symbolic model may provide a better vehicle for enhancing learning than a concrete model.

(3) The nature and type of concrete materials that lead to high cognitive and affective growth among students in primary and secondary schools (Bledsow, Purser and Frantz, 1974; Broomes, Payne and Schroeder, 1974; Harshman, Wells and Payne, 1962; Parker, Rieff and Sperr, 1971).

BROOMES and co-workers (1974) used information gained from evaluation activities to guide and motivate the construction of certain concrete experiences for pupils. Within actual classrooms, the teacher taught his class "as a <u>set of groups of six pupils</u> using teaching methods that could be described as <u>exploratory</u> to generate among his pupils cognitive behaviours such as <u>understanding</u> and <u>synthesis</u> and affective behaviours such as <u>attending</u> and <u>responding</u>" (p.72). Using ordinary articles found in the environment (drinking straws, popsicle sticks, bottle corks, etc.), pupils and their teachers came face with profound mathematics in what appeared to be simple activ-

ities.

HARSHMAN, WELLS and PAYNE (1962) compared the payoff goodness of three different types of concrete materials and found that significant differences were always in favour of materials selected and built by teachers as against commercial materials.

2.2 The Cognitive-Affective Emphasis

A preoccupation with affective factors (attitude, interest, self-reliance, enthusiasm) has developed among researchers for many reasons. First, it seemed reasonable to assume that the way children reacted to mathematics influenced the quantity and quality of their learning. Second, at those educational levels for which data are most reliable (high school and college) measures of ability account for only 35 to 45 percent of the variation in academic performance (Lavin, 1965).

(1) Therefore superior theoretical formulations of mathematical learning tended to emphasize equally both cognitive and affective outcomes (Broomes, 1971; Naylor and Gaudry, 1973).

BROOMES (1971) investigated certain cognitive and personality factors underlying mathematical ability, and supported the viewpoint that learning strategies in mathematics should focus, at one and the same time, on cognitive as well as affective considerations.

The investigation employed a theory of the nature of mathematical abilities in which personality and other factors were allowed to interact with ability factors. The results of the analyses showed a General Personality-Cognitive factor at the top of the hierarchy of factors. After the removal of the general factor, the variables tended to fall into two main groups, a Total Adjustment factor and a General Cognitive factor. This finding sup-

ported the equal importance assigned to cognitive and affective factors in classroom learning of mathematics.

2.3 The Goal Structure in mathematics classrooms

Researchers in mathematics learning had often conducted their activities under the assumption that in a classroom there existed only a competitive goal structure where the pupils were expected to outperform their friends, sometimes in an unfriendly amoral way (This seems to suggest another factor that could account for the inconsistent findings that tended to come from studies done in artificial conditions and those done in actual classrooms). During the 1970-75 period, research activities have been directed towards understanding and using the motivational forces associated with other goal structures (Johnson and Johnson, 1974).

(1) The existence of three goal structures within classrooms - cooperative, competitive, and individualistic - which control and define the social relationships among the pupils and teachers as well as the relationships among these persons as they attempt to attain goals within the classroom.

(2) Each goal structure with its unique set of events, places a powerful impress on the cognitive and affective functioning of pupils within the classroom (Clifford, 1971; Edwards, De Vries and Snyder, 1972; Paris, 1971).

CLIFFORD (1971) showed that competition may be superior to cooperative or individualistic goal structures when the task the pupil is required to do is a simple drill activity or when sheer quantity of work is desired on a mechanical or skill-oriented task that requires little help from another person.

On the other hand, cooperative goal structure results

in higher achievement than does a competitive goal structure when the instructional task is some sort of problem solving activity; or when the task requires students to depent on each other.

PARIS (1971) identified certain characteristics of students in cooperative and competitive arithmetic situations.

(3) The importance of the cooperative goal structure in learning mathematics (Broomes, Rahman and Richards, 1971; House, 1975; Steele, Walberg and House, 1974).

BROOMES, RAHMAN and RICHARDs (1975) working in actual classrooms (where a competitive goal structure was prominent) found that "teachers do not spend enough time trying to lay the foundation of basic items in a way which involves pupil-pupil and teacher-pupil interactions". As a result "...children not only encounter a number of difficulties as they move from class to class but also an aversion for the subject".

A solution worked out under the Guyana Mathematics Project, Guyana, gave pupils opportunities to cooperatively learn mathematics. Their findings showed that pupils under the cooperative goal structures undertook tasks which contributed much not only to high achievement and high activity, but also to enjoyment and group empathy.

The following two studies seem to indicate important research directions for a goal structure that would optimize learning outcomes in mathematics.

HOUSE (1975) provided evidence for the existence of certain relationships between learning environments in the classroom and academic self concepts, and also demonstrated that, for ninth grade mathematics students, both general and mathematical concepts of academic

ability can be predicted from combinations of classroom environment characteristics.

STEELE, WALBERG and HOUSE (1974) investigated the nature of the cognitive press associated with mathematics and showed that mathematics was sharply distinguished from other subjects on a convergence-divergence factor, and clustered with language arts on syntax-substance and objectivity-subjectivity factors.

3. SUMMARY

Factors such as a redefinition of curriculum development and a reinterpretation of research findings seemed to be associated with certain research activities in mathematics learning, and these activities can be put together into three clusters which may be labelled

(1) The Concrete/Semi-concrete/Abstract Movement,

(2) The Cognitive-Affective Emphasis, and

(3) The Goal Structure for learning mathematics in schools.

These clusters derive their importance from philosophical and cultural positions taken up in developing countries.

OUTLINE OF IMPORTANT RESEARCH IN THE FIELD, 1970 - 1975

Jere E. Brophy

I am not specialist in mathematics education specifically. Instead, my research has concerned the dynamics of teacher-student relationships (especially teacher expectations and attitudes), studies of the stability of teacher effects, and process-outcome studies intended to identify aspects of effective teaching. Completed work in the latter area has been done in elementary schools, although follow up work is being done in junior high schools (seventh and eighth grade). My comments will be focused on these broader areas rather than on math education as such.

1. IMPORTANT RECENT DEVELOPMENTS

1.1 Development of new and better methods, and wider use of these methods to collect data on individual students. Until recently, remarkably few data existed concerning differential teacher treatment of different students in the same classroom. These topics have been studied recently in relation to teacher expectations and attitudes as well as to student individual differences.

1.2 Aptitude treatment interaction (ATI) and related research. On the one hand, researchers in this area have conceded that few aptitude treatment interactions exist, and these few are too weak to be of much practical importance. In the other hand, interactions between learner preferences or learning styles and treatments (teaching methods) are being discovered which may have practical implications for differential instruction. These include preference for competitive versus individual versus cooperative learning situations; preference for structure versus free-

dom or learner control; preference for lectures versus discussions or projects; and many other differences discussed in the paper "Interactions Between Learner Characteristics and Optimal Instruction".

1.3 Recent improvements in several aspects of process-outcome research have led to replicated findings which have implications for teaching. There is movement away from the conceptualization of good teaching as mastery of a few "crucial" techniques towards a conceptualization of good teaching as mastery of many techniques which are orchestrated for use in optimal ways as situations and individuals require. New measurement techniques are taking into account individual differences, context differences, subject matter differences, and other differences that affect process-outcome relationships. These factors apparently were masking such relationships in previous research. This is illustrated in the books by DUNKIN and BIDDLE (1974) and by BROPHY and EVERTSON (1976).

1.4 Investigation of non-linear relationships. It seems obvious to everyone that relationships between teaching methods and outcomes are likely to be non-linear. Most typically, an inverted-U relationship is to be expected, indicating that an optimal amount of a teaching method produces better results than either not enough or too much. Despite the seeming obviousness of this observation, few investigators even have analyzed their data for non-linear relationships until very recently. However, those who did found some interesting results (see the book by Brophy and Evertson).

1.5 The development of realization that a systems approach needs to be taken in interpreting process-outcome data. This has been stressed by Lundgren with his concept of "frame factors", and it is coming to be realized by

others who are beginning ot see that certain relationships between teacher behavior and student outcomes are to be expected only in certain kinds of learning situations. Along with the tendency to take into account contextual differences, this is part of a growing awareness of the need to specify the parameters and limits of generalization that apply to findings, in addition to discussing the findings themselves. This will help eliminate some of the confusion and problems that arise when data which are valid for one context or learning system are generalized inappropriately to a different context or learning system.

1.6 Some important negative results have developed recently showing that several educational fads popular in America have been mistakes (the "new math", open education, and oral language based curricula, in particular). Data accumulate daily showing that these approaches are inferior to traditional approaches as well as to other new approaches that are based on more valid assumptions. Open education is not working for what seem to be obvious reasons: children find it harder to learn in large and noisy environments compared to smaller and quieter environments. The failure of the new math is a special case of something to be described below: assumptions about learning based upon the logic used by adults do not necessarily hold for younger learners, and they almost certainly do not hold for children who are not at least at the concrete operational stage as described by Piaget. Oral language based curricula are unsuccessful primarily because oral language is much less central to general cognitive development than cognition is. Oral language based approaches tend to reverse the priorities here, and thus spend more time and attention on secondary matters at the expense of more primary matters.

1.7 As information about cognitive development and memo-

ry organization is accumulated, it becomes more and more clear that learners perceive and organize information in their memories according to their unique interests and development levels. This can be stimulated and facilitated through teaching, but only to a limited degree. One implication here is that the idea of finding "ideal" sequences for presenting information is not feasible, because learners do not all apprehend information the same way, even if one approach is most logical and efficient. Also, data from these same areas of research are beginning to place more and more stress on the importance of active responding in learning and of application of learning through active exercises to insure overlearning and generalization. These and related data from developmental psychology and the psychology of learning are calling into question some of the approaches to instruction which rely too heavily on media or on presentation of information in ways that allow learners to remain primarily passive, and they underscore the value of traditional methods such as practice.

1.8 All of this, coupled with national assessment data indicating that student achievement on standardized tests has been dropping over the last ten years or so in the United States, is leading to an emphasis on returning to "the basics". This means different things to different people, but common elements include the need for primary attention on reading, writing, and arithmetic, and upon the application of these fundamental tool skills to problems of living. As a result, this means correspondingly less stress on other topics (moral development, values clarification, social studies generally).

1.9 An insight that has developed recently is that learning and teaching in the first few grades where most

time is concentrated on teaching the basic tool skills is different in many fundamental ways from learning and teaching later, when these tool skills are used for learning about other things (see the materials by Brophy and Evertson). Most research in classrooms has been done at the middle grades or higher, and attempts to generalize the findings of this research usually are inappropriate if they are extended to the early grades. This is especially true of indirect teaching, advance organizers, and other variables related to classroom discussions, because classroom discussions occur relatively seldom in the early grades. In contrast, relatively little attention has been given to matters that do occur regularly in the early grades, especially instruction in fast paced drills occurring in small groups (especially reading groups), presentation and explanation of the operations involved in doing seatwork assignments, and methods of monitoring seatwork and providing corrective feedback and reteaching as necessary.

2. DEFICIENCIES IN PRESENT RESEARCH AND RECOMMENDATIONS FOR THE FUTURE

2.1 Research intended to generalize directly to classrooms should be conducted in classrooms in the first place. Otherwise, differences between artificial experimental situations and regular classrooms may be so great as to prohibit the generalization of experimental findings to everyday teaching. Generalization of experimental data should be tested, not merely assumed.

2.2 Findings should be analyzed routinely for non-linear relationships (in correlational studies) and for interactions between treatments and outcomes (in studies of group or treatment differences).

2.3 Investigators should specify the limiting parameters and contexts which were and were not taken into account in the research itself and which might be relevant in generalizing the research.

2.4 More attention needs to be given to teachers' goals and philosophies. More generally, we need more cooperative research in which the teacher is a participating partner rather than a "subject" who is kept uninformed of the nature and intent of the study.

2.5 Simple frequency counts of classroom processes are not very useful. They should be replaced with percentage scores or other scores that combine frequencies with information about opportunities or expected frequencies for particular interactions. For example, a measure of the frequency of praise of correct answers is less useful than a measure of the percentage of correct answers which were praised.

2.6 Future research should take more seriously the observation that effective teaching is not a matter of implementing a few crucial techniques but instead is a matter of applying a large number of limited principles when and where they are needed. This means more restrictions on the range of classroom situations included in a given study, but much deeper and broader study of these situations. In math, for example, the methods and materials best suited to teaching introductory number concepts and computations probably are very different from those most useful for teaching subjects like geometry to older students who have mastered math fundamentals.

2.7 Much more attention needs to be given to the curriculum and to the quantity and quality of seatwork and the teacher-student interaction that occurs in relation to this seatwork. Much educational research has

focused primarily or exclusively on verbal interaction in public discussion situations. This is not the only aspect of education, and in many contexts it is not even a very important one. We are missing some of the most important contributors to learning by concentrating on public teacher-student interaction at the expense of investigation of the curriculum (what exactly is taught) and of seatwork assignments (what are the students required to practice and how does teacher monitor, correct, and provide remedial teaching?).

2.8 Although this observation has been made many times before, educational research needs more attention to teaching, relative to learning. My own belief is that research on teaching is truly primitive at present. We do not even have basic taxonomies and terminology, let alone complex theories or research paradigms. So far as I can tell, the biggest single reason for this is that relatively little research is done in real classrooms and/or in conjunction with working teachers. As a result, "educational research" often involves study of activities that bear little resemblance to what goes on in schools. The identification of relevant context differences, as discussed by DUNKIN and BIDDLE (1974), is especially important if progress is to be made here.

2.9 We also need to recognize that effective teaching is a matter of optimizing, and that this will be defined differently depending upon goal priorities. Some teaching approaches apparently produce the most efficient learning but at some cost in student attitudes towards teachers or schooling. Other approaches produce favorable student attitudes but are relatively inefficient as teaching devices. Depending upon the relative weights one wishes to assign to cognitive versus affective outcomes, "effective teaching" would be defined different-

ly here. The problem is compounded if you take into account other goals such as socialization of students concerning citizenship or morality in addition to goals more directly related to the curriculum.

2.10 Experimental studies should involve collection of process data concerning implementation of the teaching methods being studied. This should be done both in the experimental group (to see if instructions are being followed) and in the control group (so that their spontaneous use of the methods being studied can be taken into account in assessing the effectiveness of the methods and/or of the treatment).

PROBABLISTIC THINKING IN CHILDREN AND ADOLESCENTS

Efraim Fischbein

In the past decades we have been witnessing a growing concern with probabilistic and statistical reasoning within scientific thought. Consequently, in many countries there is an increasing tendency to introduce the theory of probability in mathematical curricula. The teaching of a new branch of science raises many psychological and didactical problems. For example, what is the best age to start teaching it? What should be the proportion of concrete to abstract elements, of axiomatic to intuitive presentation in order that the acquired body of knowledge should be as reliable and viable as possible?

In the following lines we intend to give a rough outline of the structure and development of probabilistic thinking in children and adolescents. We think that such a view may be useful to achieve curricula for teaching probabilities in elementary and junior high schools.

1. THE INTUITIVE BACKGROUND

The essential fact is that the spontaneous background of probabilistic thinking is a mixture of correct and incorrect intuitions. Research we recently carried out (Fischbein et al., 1971) revealed that adolescents possess a natural, correct understanding of the concepts of chance and of quantification of chances as the relationship between the number of favourable and of possible outcomes. The subjects understand intuitively that the probability of the union of elementary events, which constitutes a compound event, is obtained by addition. On the other hand, subjects do not realize intuitively the compound nature of some events and have some difficulties in determining the probability in such cases (for instance, the subject have to compare the probability of obtaining a

6-5 and a 6-6 by throwing a pair of dice. They do not see intuitively that the probability of 6-5 is twice the probability of 6-6). Concerning the multiplication law, we have noted a natural feeling of the reduction of odds when the number of imposed conditions increases (intersection of events). For instance, Ss understand intuitively that the chance to draw a black marble from a box containing a black and a white marble is greater than the chance of drawing twice the black marble (with replacement) from the same box.

On the other hand, natural probabilistic thinking is seriously affected by a grat variety of erroneous intuitions. Tversky and Kahneman have described some typical errors in judgment under uncertainty. Knowledge of such heuristics is of main importance for educational purposes. It suggests ways to prevent biased interpretations by adequate teaching methods. Let us remember the essential heuristcs described by Tversky and Kahneman: representativeness, availability, and anchoring.

Representativeness. Let us suppose we have to answer a question of the following type: What is the probability of an object A belonging to a class B? Generally, people evaluate the probability by the degree to which A is considered to be representative for B.

Let us take an example: On each round of a game, 20 marbles are distributed at random among five children: Alan, Ben, Carl, Dan and Ed. Let us consider the following distribution:

	I	II
Alan	4	4
Ben	4	4
Carl	5	4
Dan	4	4
Ed	3	4

"In many rounds of the game will there be more results of type I or of type II? A significant majority of subjects considered distribution I as more probably because distribution II appears to be too lawful and consequently non-representative for a random process. In fact the distribution II is more probable than distribution I" (Kahneman and Tversky, 1972). This heuristic appears to be insensitive to prior probability of outcomes, to sample size etc.

Availability. "There are situations in which people assess the frequency of a class or the probability of an event by the ease with which instances of occurrences can be brought to mind" (Tversky and Kahneman, 1974). Availability may be an useful heuristic for assessing frequency or probability because in many cases instances of large classes tend to be more easily recalled. But in other cases, as an effect of various factors, there are instances which are more easily recalled than others without being more frequent. "It is a common experience that the subjective probability of traffic accidents rises temporarily when one sees a car overturned by the side of the road" (Tversky and Kahneman, 1974). Or, a second example: Let us consider a group of 10 people who form committees of 2 members. How many committees of 2 and 8 members respectively can be formed? The median estimate of the number of committees of 2 members was 70; the estimate of 8 members was 20.

Subjects may more easily imagine groups of 2 by 2 people than groups of 8 by 8 people and, consequently, they consider the frequency of the first category to be greater, though often, in fact, the frequencies are equal.

Anchoring. A third heuristic described by Tversky and Kahneman which biased the objective estimation of frequencies and probability is anchoring: "...different starting points yield different estimates, which are biased toward the initial values".

Two groups of high school students were asked to estimate

within 5 seconds the value of 8. In one case the expression has been written as 1+2 ... 8. In the other case the expression has been written as 8x7 ... 1. "The median estimate for the ascending sequence was 512, while the median estimate for the descending sequence was 2250. The correct answer is 40320" (Tversky and Kahneman, 1974, p. 1128).

Fischbein et al. (1970) have described a strong tendency to underrate the number of permutations with 3, 4, and 5 elements. The underrating is more significant the larger the number of objects considered. For instance, the mean estimation for 3 elements was 5.30 (instead of 6) for 4 elements it was 9.60 (instead of 24) and for 5 elements it was 16 (instead of 120), these results having been obtained from pupils age 14-15. Consequently, any subjective estimation of probabilities, supposing an estimation of permutations, will be biased by such underratings of the number of possible outcomes.

2. THE ROLE OF MISUSE OF LOGICAL AND MATHEMATICAL SCHEMES

In many cases the errors in probability estimation may be explained in terms of the inappropriate application of operational (non-probabilistic) schemes. We quote an example from Freudenthal: "Du Méré knew that it was advantageous to bet on the occurrence of at least one six in a series of four tosses of a die ... He argued it must be advantageous to bet on the occurence of at least one double-six in a 24 toss series with a pair of dice". The result is incorrect and it has been reached by simply and inadequately using "the rule of three" (Freundenthal, p. 151-152). These facts and many others described as paradoxes of probabilistic thinking lead to the conclusion that probabilistic intuitions are naturally biased by a number of typical, inadequate heuristics. Engel goes even further. He wrote: Why has probability not had the impact on mathematics that geometry has? Because we have a natural geometric intuition

but no probabilistic intuition" (Engel, 1970, p. 88).

Things are not exactly so. In fact, we possess a number of primary natural probabilistic intuitions but they are generally neglected by science and mathematical education. In elementary and junior high schools, science education is mainly oriented to developing deterministic thinking.

3. THE DEVELOPMENT OF PROBABILISTIC THINKING IN CHILDREN

The Piagetian theory. Let us first summarize Piaget's theory which constituted the basic framework for the discussion which follows. The logical and arithmetical operations are internalized actions organized in systems. They are basically characterised by rigorous composition and reversibility. Such a reversible composition makes possible deduction and lawful predictions. Chance transformations, chance events are not regorously reversible and cannot be rigorously composed in deductive systems. During the pre-operational stage, the mind of the child is unable to differentiate between chance and necessity, simply because it does not yet possess the operational framework. Being unable to understand necessity he is unable to understand -- by opposition -- the unnecessary, the fortuitous.

During the concrete operational stage the child gets the essential intellectual instruments to understand necessity and to produce deductive reasoning. At the same time the child discovers by contrast the meaning of chance events and becomes able to distinguish them from the strictly deductive ones. But this differentiation is not enough to produce probabilistic thinking. Probability expresses the fact of absorbing chance in an operational framework, i.e., the possibility to systematize, interpret, and predict chance events by operational (logical-mathematical) means. That synthesis between the possible and the deductive, constitutes the source of probabilistic thinking. It is only during the formal operational stage that this synthesis

will occur (Piaget and Inhelder, 1951, p. 247). This is briefly the Piagetian point of view. In fact, some recent findings have to be taken into account in order to obtain a more complete understanding of the development of probabilistic thinking.

4. PROBABILIY LEARNING

Let us consider the following experimental paradigm. In front of the subject there are two different coloured lights and two keys. A third light is used as a "ready" signal. In each trial the subject has to press one of the keys corresponding to the light he predicts will come on next. At the beginning, of course, the reactions are random and therefore each of the keys has the same chance to be pressed. Afterwards the subject's predictions tend gradually to approach the frequencies of the stimuli. This has been termed as probability learning. The subject does not learn a certain answer corresponding to a certain stimuli as in ordinary conditioning. What he has to learn is the relative frequency of certain stimuli, the probability of their appearance.

The fundamental fact, described by many authors, is the phenomenon of probability learning itself, the matching behavior. But analogous to the conditional reflex which is at the same time both an important physiological phenomenon and an investigation device, probability learning is not only an interesting psychological phenomenon but also a very fruitful technique to investigate the intuitive sources of probability thinking. Probability matching behavior does not presuppose any conceptual system proper and it serves as an index of the presence of behavioral schemes adapted to stochasic conditions. Such a divice is therefore particularly useful to investigate the development of probabilistic intuitions in children. Of course, this allegation means that probability matching behavior is really equivalent to an intuitive, global

estimation of odds.

The essential finding is that matching behavior appears even in very young children (3 to 4 year olds). (Messik and Solley, 1957). The children had to predict which one of the two size values ("big" or "small") of cardboard figures will be presented by the experimenter at the next trial. The following frequency pairs were used: 1.00-0.00, 0.90-0.10; 0.75-0.25; 0.60-0.40. The subject's predictions tend to approach the input probability level. Similar findings have been described by CRAIG and MYERs (1963), SIEGEL and ANDREWs (1962), STEVENSON and WEIR (1959), etc. The majority of authors conclude that the probability matching behavior is a developmental phenomenon where the frequency of predictions approaches the probabiliy level more rapidly as the child gets older.

The essential conclusion is therefore that the global intuitions of relative frequency and probability are present even in pre-operational children. Research has shown that the functioning of that intuition is influenced by several factors which have to be taken into account in teaching situations. Rewarding for correct guesses increases the number of predictions of the most frequent event regardless of age ("the maximizing strategy"). Reward also introduces new dimensions in the choise behavior. (A simple calculation shows that the most profitable behavior, i.e., with a maximum number of confirmations is obtained by predicting only the most frequently reinforced stimulus. For instance, for the frequency pair 70% - 30% the probability matching behavior will result in a probability of confirmations equal to $0.70 \cdot 0.70 + 0.30 \cdot 0.30 = 0.58$. With the maximizing strategy the percentage of confirmations is 70%).

Two important conclusions may be drawn from this fact:

(1) The primary intuition of chance is able not only to

evaluate frequencies but also to adapt itself to more sophisticated demands such as the maximizing strategy and

(2) In probability games reward is an important factor. Reward for correct predictions may contribute effectively to the building of primary correct probabilistic intuitions.

The predictions are affected by an alternation tendency -- the tendency to automatically alternate the choices regardless of the order of stimule; (for instance, left-middle-right, etc. in an experiment with three possible responses). The maximum of that tendency appears in 7 to 9 year old children. This alternation tendency (which interferes with the probability matching behavior) expresses the natural need for order in the child. Because of the randomness of the reinforcing events, the child feels himself unsure of his predictions. He neglects the stimuli which are out of control and discovers the possibility of imposing a clear, predictable order to his own reactions. The tendency to alternate predictions is not equivalent to the absence of probabilistic intuitions. The essential fact ot be taken into account is that the child of this age is unable to synthesize two apparently divergent cognitive structures: the intuition of chance and the need for regularities (expressing his need for a deterministic, predictible order).

5. COMPARISON OF RATIOS AND CHANCES

A different research strategy has been used by Piaget and Inhelder in the comparison of ratios and chance. A typical research situation had colored marbles placed in two boxes in different proportions. In the simplest and most common experimental situation there are marbles of two colors. After examining the contents of the boxes the subject has to indicate which one of the two boxes offers him a grater chance to extract a marble of a certain

color.

In an experiment, conducted by Fischbein et al. there have been three categories of problems. In the C_1 category, the compared ratios were of the form $\frac{a}{b} : \frac{c}{b}$ or $\frac{a}{a} : \frac{b}{c}$ or $\frac{a}{b} : \frac{a}{c}$. The possibility to reduce the comparison to a binary operation facilitated the choice. In the C_2 category of trials no restriction was imposed (for instance 6/2 : 10/5). In the C_3 category the ratios to be compared were equal.

Results have shown that the pre-operational and concrete operational children evaluate and compare chances correctly only in the C_1 category of trials. The formal operational children give spontaneously correct answers to three types of problems. These findings seem to confirm the results of Piaget and Inhelder concerning the concept of proportions. CHAPMAN (1975) using a similar procedure confirmed those findings. Before the age of 12 children are unable to operate spontaneously with proportions in which a double comparison is required. They are able to solve only the C_1 type of problems in which the comparison is reducible to a binary operation.

NOELTING (1975) has studied the development of proportion reasoning in children age 6 to 16. Drawings of two sets of cups were presented to the subjects. In each one of the sets there were a number of cups full of orange juice and a number of cups full of water. The proportion of orange and water cups differed from item to item. The children had to imagine that the contents of the cups of each set separately had to be mixed. They were asked to evaluate in which one of the two obtained mixtures the orange taste was stronger (or if it was the same). The results indicated a developmental progress from pre-operational to formal operational reasoning. The findings were partly similar to our findings. The pre-operational children solve only problems in which comparison of ratios are

reducible to simple binary comparisons (for instance, 1,2 vs. 1,5). Formal operational children solve more complicated problems implying multiple relationships; for instance, 1,3 vs. 3,5 (but only 30-40% of the subjects solved them). The essential difference in comparison with our findings was that in Noelting's research concrete operational children spontaneously solve problems implying equivalence classes (for instance, (1,2) (2,4), while results of our research showed that concrete operational children are unable to solve such problems. The important conclusion which may be drawn from that difference is that the success of proportional reasoning depends on the nature of the content for which it is used. We have here an example of an horizontal decalage. The scheme of proportion (in the sense of an equivalence class) does not act spontaneously when it is applied to compare proportions of marbles and to evaluate chances. The same cognitive structure acts spontaneously when comparing the results obtained by mixing liquids. In fact, the difference is determined by the degree of intuitiveness of the content. This findings may be of a very general interest. In fact, nowadays it is an almost general agreement that the scheme of proportion is one of the distinctive features of the formal operational stage. In Noelting's research we have a situation in which the scheme of proportion (equal ratios) acts even in concrete operational children. If the content is familiar enough (directly connected with personal experience) the scheme of proportion is active even before the formal operational stage.

On the other hand, even formal operational subjects fail to solve various practical problems requiring the use of proportionality reasoning as an explicit mathematical tool, if they are not systematically trained. (Here we refer to pupils who are familiar with ratios and proportions from their ordinary school training.) In a research

carried out by FISCHBEIN (1975) 13 year olds have been asked to find the law which governs the relation between the amount of lengthening of a spring and the force which determines that lengthening. (The law of Hooke). Ss understood intuitively the proportionality of the two variables but they were unable to demonstrate it by organizing and interpreting their experimental data and by handling the arithmetical rules of proportions. The allegation that proportion is one of the operational schemes which characterizes the formal operational thinking is too vague, too ambiguous. In fact, an intuitive understanding of proportions is present even in concrete operational children. On the other hand, the explicit use of proportions as a mathematical tool in practical situations requires special training, even in adolescents. This means that an intellectual scheme has to be considered only in relation to a given content. A scheme by itself is only a virtuality. A special training is always necessary in order to incorporate the scheme in the flow of an efficient behavior, i.e., in order to adapt the schema to the specific structure of the content.

In a recent study, STRAUSS, STAVY, and ORPAZ (in preparation) tested proportional reasoning in children age 4 1/2 to 12 using sugar water concentrations. In contrast to NOELTING's (1975) research Strauss et al. analyzed the proportional reasoning into its component parts and tested their individual developmental courses. They determined that it is comprised of four aspects:

(1) a direct function,

(2) an inverse function,

(3) proportions where both the numerator and denominator vary, and

(4) an intensive physical quantity which is non-additive.

It was found that the direct function was solved by almost

all of the children; the inverse and proportions tasks were not solved until approximately age 10 1/2; and the intensive aspect was solved by very young children (age 4 1/2 - 6), was not solved by older children (age 6 - 10), and was solved by the oldest children (10 - 12). The curvliner developmental course of intensive physical quantity was understood such that very young children possess a syncretic mental organization which leads to a correct solution, the intermediate age children constructed a differentiated but non-integrated mental organization which has the amount of sugar as the salient variable which leads to an incorrect answer, and the oldest children had constructed an integrated thought structure which had proportional reasoning underlying its solution.

Another understanding for this finding is that the drop in correct answers on intensive quantity tasks may be influenced by formal schooling since this drop coincides with entry into school. This was tested by ANKORI, STRAUSS, ORPAZ, and STAVY (1975). There were two categories of children: schooled and unschooled. (All subjects were chosen from 3 areas in the Sinai peninsula). The results showed:

(1) In most items, before training, the percentages of correct answers were inferior to 50 (30 - 40%). After training the percentages increase evidently and in some cases approach 90 - 100% correct answers.

(2) In all items and in all ages unschooled children obtained higher percentages of correct answers than schooled children.

A potential explanation of this finding (offered by us, not by the authors) is that children possess a natural intuition for comparing ratios bases on their potential experience. School education introduces procedures and exigencies which are able to help only after special and adequate training. If each training for using theore-

tical models in corresponding practical situations is absent, the theoretical knowledge may deteriorate the naive intuition. The practical conclusion is the following: If mathematical education is to be really efficient, we have to include in the curricula special opportunities which would train for the flexible use of mathematical models to solve practical problems. The mathematical schemes do not work by themselves in connection with definite contents.

6.PERCEPTIVE COMPARISONS AND PROBABILITY JUDGMENTS

We have seen that pre-operational children are able to compare chances only if the comparison is reducible to a binary operation. Some authors argue that this means that the reactions of preschool children have to be interpreted only as perceptual comparisons and not as probability judgemts. This is the point of view of HOEMANN and ROSS (1971). Let us briefly describe one of their experiments. A series of paper discs were segmented in black and white with varying proportions. In one experimental condition two discs were attached to two spinners. The subject was asked to choose the spinner which would give the best chance of winning on the color designed by the experimenter. In the second condition the subject had simply to choose the disc which contained more white (or black). The main finding was that there were not significant differences between the two conditions at any age level. The conclusion was that such experiments could not differentiate between perceptual comparisons and probability judgments. Hoemann and Ross consider that such is the case with experiments with marbles carried out by YOST et al. (1962), DAVIES (1965), GOLDBERG (1966). The main conclusion was that preschool children are not capable of probabilistic judgments and their answer are based only on perceptual comparisons. Let us suppose two boxes, one containing 2 black marbles and

one white marble, the other containing 2 black and 2 white marbles. The child is asked to choose the box which will offer a better chance to draw a white marble. The pre-operational child will correctly choose the second box (which has more white marbles, the number of black marbles being equal in both boxes). Hoemann and Ross interpret that reaction as being a pure perceptual comparison and not a probability judment.

Our point of view is different. It is true that chance estimations presuppose, generally, previous quantitative comparisons (counting outcomes, perceptual comparisons, etc.). But this does not mean that the probabilistic judgment is completely reducible to the quantitative estimation. The actual wording of the test used by Hoemann and Ross was as follows: "I want you to look at the two spinners very carefully and show me which one you will spin to make the pointer point to black (white)" (Hoemann and Ross, p. 242). It is impossible to answer correctly to that question without understanding the meaning of the experience. Understanding the meaning in this case necessarily implies the intuition of chance. Our conclusion, inaccordance with YOST et al. (1962), DAVIES (1965), GOLDBERG (1966), etc. is that the intuition of chance and the intuitive estimation of probabilities appear early in childhood. Unfortunately, as we have stated above, contemporary science education fails to develop systematically these intuitions and to incorporate them in a conceptualized, explicit thinking framework. STROHNER and NELSON (1974) using a different experimental strategy found that the pre-operational child is intuitively able to distinguish between probable and improbable events. In situations where there is a conflict between verbal meaning and factual probability the last one is stronger. The child receives two puppets (or toys) and is asked to represent the event expressed by a certain sentence (for example, "The girl feeds the baby" [probable] or "The baby feeds the girl" [improbable]).

The 2-3 year old child does not follow the meaning of the sentence and produces the situation which is more probable (This means that in both cases "the girl will feed the baby"). The essential finding here is not, of course, the fact that the child does not understand the meaning of the sentence, but the fact that he is able to evaluate (in an intuitive, practical manner the probability -- or the improbability -- of a situation. Five year old children -- as Strohner and Nelson report -- appeared almost always able to rely "upon the syntactic information in sentences rather than upon strategies, falling into significant numbers of errors only when both passive voices and improbable events were involved" [Strohner and Nelson 1974, p. 670]). By 4 years olds the results present a transitory aspect. We have here a striking picture of the incrasing capacity of children, during their intellectual development, to rely upon abstract-verbal information but, at the same time, to neglect the actual probability of events.

7. THE EFFECTS OF INSTRUCTION

7.1

The concept of probability and probability calculations implies generally the possibility to operate with proportions and combinatorial analysis. In Piaget's view these cognitive schemes energe only after the age of 12 (the formal operational stage).

Is instruction capable of accelerating this process? In an above quoted experiment (Fischbein et al., 1970) we have seen that preschool and 9 year old children are unable to make probabilistic estimations when a double comparison is required. In a different experimental condition subjects of the same age were adequately instructed to compare chances. The children were shown a technique of grouping which would permit them to solve the problems.

For instance, let us consider two plastic boxes with the following composition: (4 black, 2 white) and (8 black, 2 white). By grouping the marbles in the second box we abtain (4B, 1W + 4B, 1W). i.e., the odds are equal in the two boxes, since out of 5 marbles 4 are black in each box. Assuming there were 4B/1W and 9B/2W, by grouping it was possible to bring out the surplus of black marbles in the right box. Preschool children did not benefit from that instruction. There were striking improvements in the responses of 9-10 year old children. They became able to carry out the double comparison required for the evaluation of chances and, in general, to operate correctly with the concept of probability.

7.2

FISCHBEIN et al. (1970) have shown that by using tree diagrams, even 10 year old children can rapidly understand the concepts of the arrangement and permutation and solve simple problems implying combinatorial calculations.

7.3

OJEMANN et al. (1965, 1966) and KELLER (1971) have shown that preliminary instruction has a positive effect on probabilistic behavior. In the Ojemann experiment pupils received a preliminary instruction programme during which they were familiarized with the concepts of risk, chance maximization, and prediction. Experimental subjects who were 10 year olds, in comparison to a control group, showed an increased accurary of predictions in trials with known proportions. Their results were better even in probability learning type experiments (this fact was expressed in the maximization behavior).

7.4

A large scale experiment concerning the teaching of probabilities in junior high schools (20 classes) has

been organized by us in Bucharest (1971-1974). The concepts taught were:

(1) Chance, probability, relative frequency

- Events -- certain possible, impossible events
- The concept of chance, the estimation of odds
- The intuitive comparison of odds in practical situations
- Probability as a measure of chance. The computation of probability in simple situations
- The concept of relative frequency, the stability of relative frequency
- Probability and relative frequency
- The concept of sample.

(2) Combinatorial analysis

- The set of all possible outcomes of an experience, the sample space
- Arrangements with repetition. The tree diagram
- Simple and compound events
- Arrangements without repetition
- Permutations
- Calculation of probability in situations in which the number of elementary events has to be determined by combinatorial procedures

(3) Calculation rules

- Operations with events. Union and intersection of

events. Mutually exclusive events. The additive law for mutually exclusive events.

- Dependent and independent events. The multiplication law for independent and dependent events. Conditional probability.

The teaching of these concepts was constantly based on practical activity carried out by pupils themselves (with marbles, dice, etc.). Our point of view was that, primarily, the intuitive background has to be built. Each intuition is a tool of mental and practical activity and can be constructed only as a result of personal, actual activity. Such an activity has to be carried out systematically, mainly during the concrete and formal operational stage.

The results of the teaching experiment have demonstrated the great interest and receptivity of children and adolescent (10-15 years old) for that branch of mathematics. They showed a good understanding of probabilistic concepts and the capacity to solve problems in that area in a systematic manner. By learning probabilities the pupils become more aware of the realistic aspect of mathematics, of the practical meaning and usefulness of mathematical knowledge. As Glaymann and Varga say: "...de nombreuses personnes ignorent que la mathématique touche des domaines bien plus proches de la vie courante que ceux qui sont traditionnelement enseigné à l'ecole. Ainsi, plus nous introduisons tôt des notions de probabilitiés et moindre sera le risque de croire que la mathématique est coupee de vie de tous les jours". (Glaymann and Varga, 1973, p. 10). Our teaching experiment has demonstrated that even pupils with low results in learning mathematics improve their general mathematical activity as an effect of learning probabilities.

As for the strategies of teaching probability at the elementary level I agree with the following view expres-

sed by Dietger Heitele, "The major task of didactics of probability in the next few years is not writing deductively well structured chapters on stochastics for grade level n-probability as an abstract mathematical domaine -- or to troduce still more play activities in no systematic setting. Rather than this it is necessary to integrate stochastical activities as early as possible with activities in arithmetics and even more, in geometry, and in all cases to respect and to develop meaningful connections with reality, with the world of the pupil". (Heitele, 1975, p. 203).

8. CONCLUSIONS

The general conclusions of this review are the following:

- Human beings possess a natural intuitive understanding of chance events. Research has shown that the intuition of relative frequency and probability is active even in 3 to 4 years of age children.

- An intuitive understanding of proportions is present in concrete operational children but the spontaneous use of that cognitive schema depends on the nature of the content to which it is applied.

- By using adequate techniques we may teach even concrete operational children (10 year old) simple combinatorial procedures, explicit proportional reasoning, and the concept of probability as a measure of chance.

- Children and adolescents (10-15 years of age) are very receptive to probability concepts, and related techniques of calculation. Even those children who had previously been weak in mathematics were attracted by this type of activity. Furthermore, this produced an improvement in all their mathematical abilities.

- The absence of probability concepts in elementary science

education results in an impoverishment of the intellectual capacities. The presumed synthesis between the necessary and the possible -- which is the basis of probabilistic thinking -- does not in fact take place spontaneously at the formal operational level because of the unilateral nature of science education which emphasizes only the deterministic aspect. The intuitive background of probabilistic thinking in adults is generally distorted as a result of that fundamental lack in science and mathematical education.

- We have to distinguish between virtual and actual cognitive schemas. A certain cognitive schema -- such as reversibility, proportionality, combinatorial capacity -- becomes completely effective only in connection with a given content and as a result of specific training aimed to build intervening accomodative schemas.

RESEARCH RELATED TO THE MATHEMATICAL LEARNING PROCESS

Stieg Mellin-Olsen

1.

The 1960's furnished good prospects for mathematics teachers who saw the need for a total revision of their teaching. Ideas began to circulate on how to structure syllabi so as to avoid the futile conceptual contradictions which occurred so frequently on the old ones, and it became increasingly clear to teachers how these concepts and structures could be conveyed to pupils. Either by breaking them down step by step in a Skinnerian way, by developing constructive exercises in the manner of Dienes, or by following other models of how learning of mathematics can most effectively take place.

2.

The late 1960's and the early 1970's have, however, brought us several disappointments. The mixed experience gained from special programs like the Head Start in the U.S., the general expansion in programs of special education, the daily experience of mathematics teachers ("What is the point of all this?") - all these tell us that the existing learning theories, based solely on a psychological foundation, describe only some necessary - but not sufficient conditions to be satisfied for successful learning to take place. The main argument in what which follows is that the existing learning theories in the field, are of a mechanistic, nondialectical kind in relation to the learner as an individual. They treat the individual as a passive recipient who only thinks about mathematics in line with the goal of the instructor rather than <u>one who thinks about the situation in which the instruction takes place, and chooses his strategies for learning accordingly</u>.

Thus the anthropological aspect of learning is lost sight of the fact that the individual himself ultimately decides whether learning is to take place or not. This decision depends upon a series of relations between the individual, the environment, and the school which is part of this environment. I shall sketch a few of these relations, the study of which has proved indispensable for those who wish to improve as mathematics teachers. I will subsequently sketch a few general implications for future research into the field.

3.

It is no use discussing mathematical learning, or any learning at all, without making it quite clear in what context the learning is to take place. In what follows, I will relate the problem of learning to the context of the compulsory school, at both primary and secondary level.

Such schools organise the educational setting in such a way that the syllabus to be pursued comprises a body of knowledge which will be the same for a large group of individuals, within a county, a nation, or a group of nations. This principle has been reinforced, through the introduction of the comprehensive school (Gesamtschule) at the secondary level in many countries.

This is where a seemingly paradox emerges. This type of school, based on the idea of democratizing education, creates quite large groups of lagging pupils who need remedial teaching to keep up. In other words, the new school reform which was meant to enhance the educational prospects of underprivileged groups in society, leads to an increasing demand for special education. The resultant educational policy is based on the compensation thesis: the pupil who does not succees in school, frequently comes from a home which is "culturally deprived", and school is to redress the balance.

The compensation thesis has been met with increasing resistance during the last 5 years, because research has shown it to be quenstionabel as an educational principle. I will give some examples which being typical of the problem, demonstrates this point quite clearly:

> "The educational administrator in the Indian reservation school, describes the difficulties of educating the Sioux Indian child: "The school got this child from a conservative home, brought up speaking the Indian language and all he knows is grandma. His home has no books, no magazines, radio, television, newspapers; its's empty! He comes into school and we have to teach him everything ... The Indian child has such a meagre experience".

Norway has a minority group, the Lapps. They have their own language, their own means of livelihood (reindeer), some of them are nomads. The children have to attend a "Norwegian" school. Some of them have to attend boarding schools from the age of 7. They have to learn the Norwegian language, and all the teaching will be done in it. The Lapps are politecally divided into two groups: one group whose members accept the increasing Norwegianizing and wants it to be as effective as possible, and another group whose members are resistant. In this latter group it has been discovered that the members saw it as an advantage if their kids did badly at school. This idea, so heretical for the educator, rests on a definite logical foundation: success at school increasing the danger that secondary school (or subsequent university) education will take the child away from the parents.

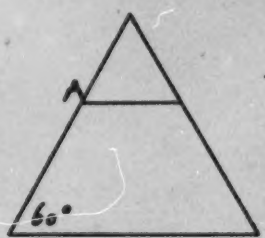
All over the world there are cultural minorities which fall victim to an expanding school system, and which clearly have their own culture, their own language, their own body of knowledge, which is not taken account of in the school. On the contrary, the school frequently defines such pupils as stimulus-deprived and in need of supplementary treatment. Bruner in discussing this, summarizes research on the culture of poverty:

"Make no mistake about it: it is a rich culture, intensely personalized and full of immediate rather than remote concerns. The issue is certainly not cultural deprivation, to be handled like avitaminosis, with a massive dose of compensatory enrichment".

5.

The problem is however, more general than that of the minority groups. It concerns the whole contradiction between the working class culture, and the one represented by school. It is reflected in all existing sociological data on school performance and social class, especially when it comes to "under-achievement", "illiteracy", special education etc., which are always mainly connected with the working class.

My neighbour's son is a carpentry apprentice. His father is a road worker. The boy is now 20. He has'nt very much good to say about school, which he left as early as he could. His knowledge of arithmetic was near zero when he left. He was built a common garage. Can you picture us constructing the roof, the rafters? The paralles? The isosceles triangels? All congruent, in parallel vertical planes, equidistant. Can you imagine the calculations when we saw we were running short of timber? Where could we save, and how? (We dicided to avoid making the triangles isosceles!) Can you see the same boy in a mathematics lesson at school:



Teacher: How large is ∠ A?
Pupil: It is 60°!
Teacher: Yes! Why?
Pupil: I can see it!
Teacher: You have to explain!
Pupil: I can measure it.
Teacher: No no, in mathematics we do not measure.
Pupil: ? ? ?
Teacher: Here are corresponding angles between parallels. Don't you remember?

The carpenter or the roadworker, has a theory about what he does. This theory is, however, related to his daily work, and is not formalized into mathematical models. The mathematician on his side, never makes the transition from model to actual performance, but keeps to the formal level, at which the concrete is merely "represented". When the carpenter and the mathematician built the garage, the job of the latter was to hand over the tools, and carry the planks. In highly industrialized countries there exists a distinct division of labour. There are those who carry out the conrete work as industrial workers, craftsmen etc., and those who work with formal representations of these, as architects, economists, engineers, teachers etc., without taking part in the first kind of work. According to an increasing body of research, it becomes more and more clear that this division of labour creates different knowledge cultures: different dialect, different structuring of language, different vocabulary, different leasure activities and so on. All this creates immense problems for teachers, when school goes comprehensive and at the same time opens its doors for the worker's children, <u>without</u> revising its syllabuses and teaching. The compensation thesis is bound to fail for at least two reasons, both of the greatest importance for learning theorists:

(1) Given the assumption that there exist different knowledge cultures in different groups of individuals, it is not sufficient to compensate for one culture relative to the other for a part of the day (as in school). The pupil still remains under family influence the rest of the day.

(2) If non-learning is produced by factors in teaching due to different knowledge cultures, the compensation thesis opens the way to reinforcement of such factors through these very efforts to compensate, that is: one risks an exponential negative effect

through application of the thesis.

Support for (2) is found in a huge body of Scandinavian research, which reports that on an average it is a negative thing for a pupil to be selected for special education, that he would have been better off if he could continued in the class.

6.

I have focused on some problems which difinitely must have some influence on how the child and his famiy think about school and thus influence learning. Through our research in Bergen among working-class families (we had long night-talks with 45 families) we found that they were quite concerned about the cultural discrepancies here mentioned. They had, however, another concern which is important for learning theorists: they were dependent on school as we all are; school was the means of getting further education, of securing the good jobs. Many families found good reasons to conceal such handicaps as word blindness from the school authorities for as long as possible. Many found it only reasonable to refuse special education for the child. The reason was clear: such a selection would destroy the pupil's later chances on the job market.

School has an objective role for the families, as an institution which qualifies children for those jobs which in turn will help them to survive from day to day as adults. With this in mind, it might well happen that the pupils have other strategies for learning than those their teachers would expect.

You can explain a mathematical principle in two ways. You can either explain it by referring to its application in a concrete situation, or you can refer to its relation to a mathematical structure (i.e. the mathematical proof). The first kind of explanation I call an instrumental explanation, the second a relational.

Norwegian pupils, aged 15, selected among the best 50% of the population, respond to a given set of mathematical principles in an instrumental way in 2/3 of the cases. Furthermore, when asked which explanations of their own they were especially satisfied with, they selected differently. 15 - 20% of the pupils selected excluselively instrumental explanations and equally many selected relational explanations. I tested both a "traditional" and a "modern" curriculum. Together with Professor R.R. Skemp I made a multiple choice version of the test, which was used on several projects in England. Although profiles vary from syllabus to syllabus, it is the similarity, the steady proportion of instrumental explanations, which is most conspicious, and which demonstrates that the pupils must have certain strategies for their learning. These strategies must of course be related to the incidentals of the teaching situation (large class, large syllabus to be covered within a certain time, traditional textbooks etc.), but it is clear that the pupil must do some common sense thinking about his learning, and this I relate to the role school plays for him. Subjects are thus not so much valued for their contribution to understanding, as for their part in getting the pupil a better job.

The learning strategies of the pupil will be dependent on school: how do we evaluate the learners performance in order to select for further education: do we favour the ability to apply mathematics, or the ability to reason? Is it the dilligence and good behaving of the pupil we favour? Is it the ability to solve problems? The pupil will soon know, and adopt a corresponding policy. What about conditions of survival in the environment? Are there no jobs for the youngsters? Are the jobs of a kind where little education is necessary? Is the town short of continuation schools? Question of this kind have also to be investigated, in order to get a picture of the pupil's total situation.

7.

Any pupil belongs to a society within which he has to survive. The character of this society, materially and politically, gives the conditions of survival. School, mathematics, education in general, has a place and a role in this. The argument is that it is not possible to say anything about learning, as long as one does not say anything about the context in which the learning takes place. Behavior in general is related to the environment of which the individual is a part, and this environment determines the conditions of survival. People may be politically divided on the question of how to survive, as the Indians, the Lapps and the workers are. The learning theorist must have all this in mind, if he wants to make his theories valid, in the sense that they "work" in the classrooms. If not he is bound to fail, as did those who tried to teach the "Lapp-minded" pupils. The argument therefore, is that in order to do research on learning, and give advice on the same, we must include the learner as an individual related organically to its environment. We need for instance to know how he and his family experience school, how they define it, and what role it place for them.

8.

Research into learning has hitherto focussed on the learner only with respect to how his existing mathematical knowledge relates to what has to be learnt. We have seen nice models of how a syllabus should be structured, and how teaching should be designed, to bring about a successful learning. Practising teachers have adopted such theories and ways of thinking, and put them into practise - sometimes successfully, sometimes, as I have pointed out, with anything but success.

The main problem for learning theories in the future is not on the microlevel, asking how different topics should

be structured in order to facilitate learning, and how the pupils should work with them in order to learn as fluently and as effectively as possible.

The main problem will be of an epistemological character. The conceptual framework in which the learning theorist works, has to be so developed as to include the learner in his individual totality, someone able to think about, able to evaluate and judge, the situation of which he or she is part. It is this pupil the teacher meets in the classroom. If the teacher is left to take over from research conducted without this grasp of the pupil, it will be a matter of chance whether he or she succeeds.

9.

I do not say that the learning theorist is going to solve all the concrete problems which any teacher faces. There will always be random conditions on the context confronting any given teacher.

The conceptual framework with which the teacher is provided in the teachers' colleges, during his follow-up education, and in the official curricular schemes etc., should however, be of such a kind, as might be applied to his particular working situation, rather than one which totally overlooks it as is often now the case.

RESEARCH RELATED TO THE MATHEMATICAL LEARNING PROCESS

Doyal Nelson

The dominant body concerned with research in mathematics learning in North America is the National Council of Teachers of Mathematics (NCTM) with head office in Reston, Virginia. Since Canada forms one of the seven organizational regions of NCTM much of the Canadian research is reported through the publications of that body. Proximity and a common language are additional factors which tend to unify the purposes of mathematics learning research in the two countries. This discussion, therefore, will apply to research being conducted in the United States as well as in Canada.

In 1970 the NCTM introduced a new periodical called the Journal for Research in Mathematics Education (JRME). It is a quarterly publication which includes reports of empirical studies in mathematics education and articles about research in mathematics education including theories and models for mathematics learning. Also within its domain are summaries of larger research projects and critiques of research projects.

There is no other journal which would reflect the research interests of mathematics educators in North America to the extent that JRME has come to reflect them. Over the years since its inception the studies published might be grouped in the folowing categories:

(1) methods of instruction

(2) achievement, skill, aptitude, cognition, problem solving and other characteristics of learners,

(3) teacher characteristics and preparation of mathematics teachers.

(4) replications or extensions of Piaget's investigations.

The dominant research design of most of the studies reported in JRME would fit a classical statistical design. It is common for researchers to use analysis of variance or covariance for treating data. A number of the studies reported were conducted as part of the requirements of an advanced degree in mathematics education. Although these studies were conducted under the direction of seasoned researchers many of the people doing the research would have very limited experience with the techniques of research and it is possible that the research reported is the first and last they will ever do. Whether or not all researchers in mathematics education are competent observers of child behavior, there is some evidence of faulty research designs.

The Center for Science and Mathematics Education at the Ohio State University with the cooperation of ERIC began publishing a quarterly journal called: Investigations in Mathematics Education. In this journal are published abstracts and critiques, by recognized active researchers, of studies in mathematics education reported in research journals. The publication of the journal is somewhat behind schedule but it is possible with the issues that exist to categorize the criticisms that the abstracters made of the studies involved. In the 1972, 1973 and 1974 issues nearly one hundred seventy-five studies in mathematics education were abstracted. Nearly four out of five of these studies were criticized either because of some flaw in design and/or because insufficent information was supplied by the researcher to make it clear precisely what procedures had been used. In about one out of ten studies abstracters noted that some conclusions seemed to be not supported by the data.

Despite the wide acceptance and use of classical statis-

tical methods for mathematics education research there seems to be some question as to whether or not these methods ban be effectively handled by some of the people reporting research. One would think, too, since replicability is such a basic requirement of scientific research, that editors would insist on at least enough detail for proper replication.

The existence of a publication whose purpose it to provide informed criticism of research studies in mathematics education is an indication of a strong desire on the part of the mathematics education community to improve the quality of its research. The nature of the criticisms would suggest that research studies shoud be more carefully designed, should be reported in more detail and that care should be taken by those reporting research to resist making unwarranted conclusions.

A promising development taking place in North America is the establishment of centres whose purposes (among others) are to conduct research in mathematics learning, to provide forums for directing and discussing research projects, and for coordinating and disseminating information about research. The most effective of these in meeting the purposes outlined above is the Centre for the Study of Learning and Teaching Mathematics at the University of Georgia, Athens, Georgia. The director is Leslie P. Steffe. In the spring of 1975, for example, five research workshops, spaced over a period of a few months, were held which attracted researchers in mathematics learning and related disciplines form all parts of the United States and Canada. The workshop topics were as follows: teaching strategies, number and measurement, space and geometry, models for learning mathematics, and problem solving. A copy of the workshop announcement is enclosed to provide information about their organization and composition. The mathematics education group at the University of Alberta in Canada

is loosely affiliated with the Georgia Center as are various other such groups in the United States.

Another centre in the United States which has somewhat similar purposes is the University of Wisconsin (Madison) Research and Development Center for Cognitive Learning. As its name suggests, one of its purposes is to apply the results of research conducted at the centre and elsewhere.

Good progress was made during the 1970-75 period in our understanding of how mathematics is learned in how children behave when involved in mathematical situations. In the other hand, even though much research energy was expended in those directions, not much progress was made in determining what teaching strategies would lead to optimal mathematics learning or in determining what programs provide the best preparation for prospective mathematics teachers.

There is need in North America for development of research procedures and techniques which will reveal how individual children acquire mathematical concepts and ideas. An encouraging trend seems to be a greater dependence on the individual interview and increased attention on detailed observation of behaviors of individual chidren in mathematical situations. This procedure is often accompanied by the use of voice or video tape to permit careful but leisurely analyses of child responses in such situations. A good example of the trend is the study: Some Computational Strategies of Seventh Grade Pupils reported by Frances LANKFORD (1972). The painstaking analysis of pupil responses to computational situations resulted in a set of conclusions and recommendations which are of basic significance to mathematics teachers.

Over the period of interest, mathematics education research developed a clearer theoretical base. As background infor-

mation accumulated researchers found it easier to select significant topics to investigate and could more readily relate the results to existing research.

ANSWER FOR MATH CONFERENCE REPORT QUESTIONNAIRE

J. Scandura

1.

In the United States, research on mathematics learning tends to fall into five general categories:

1.1 Experimental research internal to the field of mathematics education

This research is conducted primarily by people who are mathematicians and/or educators first and behavioral researchers second. In comparison with earlier research of this type (e.g., 15-30 years ago), much of this research tends to utilize relatively sophisticated statistical methods. Unfortunately, however, the questions being asked are often only symptomatic and tend to obscure the basic issues. Thus, for example, while there is a growing interest and concern for the role of heuristics in mathematical problem solving, the experimental methods of analysis used to test various hypotheses tend to be commensurate with the hypotheses under consideration. In particular, heuristic/algorithmic theories of mathematical behavior are basically deterministic in nature, and yet almost all of the empirical work is more appropriately applied in testing statistical/stochastic theories. This is a fundamental limitation, and until steps are taken to overcome it, research findings can never be expected to say anything definitive about the processes individual students use in learning mathematics.

1.2 Research by academic and educational psychologists

This research tends to fall within the confines of contemporary information processing (cognitive) psychology

and some of it is being conducted by individuals who have a reasonably strong (if not broad) mathematical background. Unfortunately, however, very little of this work is being done by individuals who have both these skills and also a broad intuitive feeling for the problems of education.

1.3 Clinical research

There is an increasing amount of "research" in mathematics education that is based primarily on observations of children as they interact with mathematical tasks of various sorts. While such research (informal observation) can be an important source of insight, especially for researchers entering the field for the first time, it all too infrequently is recognized that this approach cannot in and of itself lead to definitive findings. Intuitions and other hypotheses arising from such observation must necessarily be subjected to careful empirical evaluation.

1.4 Artificial Intelligence and Computer Simulation

This research emphasizes the role of both content and cognitive processes. While most of the research is not on mathematical learning, strictly speaking, the methods of protocol analysis used as well as the procedural representation of competence have important implications for understanding mathematical learning. A major limitation of this research is that it does not deal with individual differences in a way that has useful relevance for education.

1.5 Structural learning

Research in this tradition is very much cross disciplinary and attempts both to retain and extend the promising features of the above. In our own research on mathematical learning, for example, we have tried to combine

intuitions about the instructional process with a rigorous treatment of content, cognition, and individual differences, all within a unified theoretical framework. This research also relies on new methodological approaches that are more appropriate for testing deterministic theories. As an example of the progress made possible by these means, consider mathematical problem solving. Much of the current work in this area in categories a. and c., for example, is at the point of demonstrating that teaching heuristics can have a generally beneficial (average) effect on student problem solving behavior. Nothing, however, is said about which heuristics are good for solving what problems, nor is anything said about how to determine what heuristics individuals know when they enter an instructional situation.

These latter concerns have been resolved within the structural learning framework. Not all heuristics have been identified or evaluated, of course, but general methods have been developed which can be used to identify new ones, and existing ones have already been identified and confirmed empirically. Moreover, it is possible to determine ahead of time whether or not a given heuristic will be of value in particular problem situations, and whether individuals already have such heuristics available to them.

2.

There have been very few books on research in mathematics learning during the past five years that I would consider particularly noteworthy. Most of the best work has appeared either in books as chapter or in journal articles. About the only full length books in English of which I am aware that deal largely with mathematical learning in a scholarly way, are my own book on structural learning and that of Skemp (Psycho-

logy of Learning Mathematics. Baltimore, Maryland, Penguin, 1971). The latter book, however, only barely qualifies as a scholarly treatise and it does not have the redeeming features of novelty such as some of Polya's and Dienes' earlier books.

3.

In the United States, the most important work on mathematical learning is being done by individuals, rather than by institutions. Over the past two to three years, the National Science Foundation has funded some research at the University of Pennsylvania, the Florida State University and the University of Georgia. Only a few of the individuals involved, however, have a significant "track record" in research, even fewer have contributed to the solution of fundamental problems in mathematical learning. By virtue primarily of the presence of Pat Suppes, Stanford should be listed as one of the important institutions although Suppes work in recent years has moved somewhat away from mathematical learning. Zoltan Dienes has moved from Sherbrooke to Winnipeg, Canada and almost certainly will bring with him an active clinically-oriented mathematics education research program. The University of Pennsylvania and the MERGE Research Institute should probably also be listed in this category. The University of Wisconsin and the University of Pittsburgh also receive a considerable amount of federal money for research on mathematics and other subject matter learning. Most of this work, however, tends to follow major developments elsewhere, rather than to lead them.

PSYCHOLOGY AND MATHEMATICS EDUCATION REVISITED: 1976

Lee S. Shulman

During 1968, Professor Edward Begle invited me to prepare a chapter on the psychology of mathematics instruction for a yearbook which he was editing. I protested that, while I had been engaged in studies of teaching and learning generally with special reference to problems of learning by discovery, I could claim no special expertise in the area of mathematics instruction. Begle responded that this was perfectly acceptable. Motivated more by temerity than wisdom, I proceeded to prepare such a chapter which appeared in the subsequent NSSE Yearbook on Mathematics Education (Shulman, 1970). In the ensuing years the chapter has been characterized in many ways, most accurately as an exposition which dealt far more with psychology than with mathematics, a flaw clearly attributable to the limitations of its author.

In the years which followed, I have been examining principles of teaching in the natural sciences and medicine. With a group of colleagues, I have been most vigorously conducting empirical and theoretical studies in the mysteries of medical diagnosis and decision making. It was thus a surprise (and a bit anxiety-provoking) to receive the invitation to participate in this working group. The collaboration of Ms. Janet Shroyer, who has maintained continuing involvement in mathematical learning and teaching, significantly reduced that anxiety. Thus, with this disclaimer regarding my prolonged absence from the problems of mathematical learning. I shall proceed with the agenda of this brief exposition.

I intend first to summarize the major propositions of the paper I wrote nearly eight years ago. At the time, it represented for me the major psychological influences on the problems of mathematics curriculum and instruction.

Both its emphases and its serious omissions reflect the particular obsessions of educational psychology during the middle 1960's. I shall then analyze and outline the paper I would have written had the commission from professor Begle come in 1976 rather than 1968. It is difficult to distinguish how much of the change is attributable to developments in the field of psychology and how much to metamorphoses in this particular psychologist, but I shall gladly bequeath that probelm to historians or psychoanalysts.

The earlier chapter selected a single instructional issue as the basis for dividing the field into its major constituents. That issue, which had been a significant interest of mine for several years, was learning by discovery. Two eloquent advocates of guided learning -- Robert Gagné and David Ausubel -- were contrasted in their positions on the learning of mathematics with two equally assertive defenders of discovery learning -- Jerome Bruner and Jean Piaget. Employing somewhat stark caricatures of the opposing positions, the underlying assumptions, principal emphases and instructional implications of these views were explored, with particular emphases upon the positions of Gagné and Bruner, each of whom had influenced approaches to mathematical learning in the United States rather directly. The parochialism of the analysis was that of both psychology and North America. This dual parochialism will be equally evident in the present manuscript.

In the 1969 chapter, the neo-behaviorism of Gagné was contrasted with the cognitive developmentalism of Bruner. Although Bruner typically cited the work of Piaget as evident in support of his pedagogical assertions, a rift had emerged between Geneva and Cambridge. To this outside observer those disagreements had few educational implications. They sounded very much like members of the same church, praying in adjoining pews while debating over details of theology or ritual. Ausubel was a bit of an anomaly, since he

insited on thinking about learning in a cognitive manner, while advocating teaching in a highly guided way. Nonetheless, the general framework of the chapter treated school of psychology as a set of starting points for educational theory, moving a generation of psychologists nursed on the debates among Thorndike, Skinner and Hull, among others, to a more contemporary concern for their modern replacements. The emphasis of the exposition was exclusively on the influences of these varieties of psychological theory on the development of curriculum or the organization of instruction and learning. Where was the teacher in this classroom so fully populated with curricula and learners? Alas, he was nowhere to be found.

In the period since the late 1960's significant changes have taken place in both psychology generally and the psychology of instruction in particular. A new mainstream has emerged to replace the older battleground of behaviorism and cognitive psychology -- information processing theory. This variation of cognitive psychology has emerged as the dominant theoretical view of most educational and learning psychologists. Most dramatically, Robert Gagnè, whose writings could be suitably employed to represent educational neo-behaviorism in the earlier chapter, has now announced himself as espousing an information-processing view (Gagnè, 1974). His primary antagonists, Ausubel and Bruner, are no longer active in educational research. A new generation of educational information-processing psychologists has emerged to replace them.

A second important change that has taken place since the late 1960's is a renascence in the field of research on teaching, rather than exclusively on learning. Moreover, there is a current movement toward applying that same cognitive information processing approach to the phenomena of teaching as have been applied to the processes of learning, perhaps presaging an eventual, though still

remote, unified information-processing theory of teaching and learning.

In the next section, I shall briefly review the history of information processing theory and its impact upon psychology and education. We shall then examine several examples of research in this new tradition as it is conducted in the study of mathematical learning. These investigations will be discussed as examplars of both the concepts of information processing psychology and its characteristic research methods. Finally, we shall sketch the general features of an emerging information-processing psychology of teaching.

1. ROOTS OF INFORMATION-PROCESSING THEORY

In the middle of the 1950's, a series of publications emerged which, though produces quite independently, heralded a revolution in contemporary psychological theory. With little eleboration, these are some of the events which contributed to the creation of a totally new Zeitgeist in American psychology.

- BRUNER, GOODNOW and AUSTIN (1956) publish a research monograph entitled A Study of Thinking which combines the attempt to characterize strategies of concept learning with applications of decision theory to discuss problems of memory limitations and cognitive strain. Use of the naughty word "thinking" in the title is noteworthy.

- George MILLER (1956) discusses the "magic number seven plus or minus two" as an estimate of the limitations of short-term memory, introducing notions of bits and chunks to the psychology of learning and memory.

- Noam CHOMSKY (1957), a young linguist at M.I.T., publishes the remarkable Syntactic Structures, the first systematic presentation of transformation grammar and his theory of linguistic competence, a theory destined

not only to rock the field of lunguistics, but that of psychology as well.

- Three behavioral scientists at Carnegie Tech in Pittsburgh produce a computer simulation capable of proving theorems from Whitehead and Russell's Principia. NEWELL, SHAW and SIMON's (1958) problem solving programs not only imitate human intelligence, but even invent theorem proofs that are novel. In their work, they lay the ground rules of using computer simulations to test and refine theories of human information processing.

- These individual books and papers were finally summarized into a single theoretica position regarding thought, action and language. This general statement of the information-processing position was Plans and the Structure of Behavior by MILLER, PRIBRAM and GALANTER (1960). It continues to serve as a significant theoretical statement today, more than a decade and a half later.

- The work of Jean Piaget had long antedated the cognitive information processing revolution of the mid-50's, but had never before taken root solidly in the soil of American Psychology and education. The cognitive information processing revolution created an intellectual climate which made the reading of Piaget on the North American continent not only possible, but influently and meaningful. And in 1961, J. McVicker-Hunt's book, Intelligence and Experience, demonstrated that Piagetian theory and information processing theory could be synthesized into a single perspective on the role of experience in the development of intelligence.

Ironically, while these cognitive concepts had a profound impact on the kinds of mathematics and science curricula produces during that period, lending credibility ot the notions of problem solving, discovery and early readiness which permeated those efforts, experimental research on

learning, thinking, and teaching remained rather traditional and neo-behavioristic, (It is worth noting that during this period the principles which dominated the approaches to instructional evaluation, classroom management and environmental design were highly behavioristic, even Skinnerian. Thus, elementary and secondary school curricula emphasizing discovery and conceptual invention were evaluated using behavioral objectives and task-analytic reductive frameworks in classroom managed by teachers employing behavior modification.)

The cumulative impact of this revoluation in psychology is perhaps best expressed by one of the leading spokesmen of behaviorism, Gregory Kimble. In reviewing the most recent edition of that bible of learning theory, Hilgard's (and Bower's), Theories of Learning, he remarks on the changes in his discipline,

> "...how far we have come in the past ten years; that the white rat and the pigeon no longer provide the majority of our data, that complex mazes are rarely used these days, that S-R has been deposed as king of the theoretical hill, that 'mind' is no longer a dirty four-letter word, and that petty bickering over small points in little theories no longer describes the standard mode of intellectual interaction among the psychologists of learning..." (Kimble, 1975)

Having established the transition of American psychology from neo-behaviorism to cognitive information processing, we can illustrate the shift through briefly describing the changes in one psychologist's position. Robert Gagné has become closely identified with a perspective on the conditions for learning that treats all forms of learning as hierarchically organized in relation to one another, with classical and operant conditioning as the foundation for the hierarchy and problem solving as its peak. Any form of learning could be reduced to its components, which were sub-tasks, and these could be systematically taught in a guided fashion to ensure learning. Over the past ten years there has been a metamorphosis in that

position, transforming a neo-behaviorist into an information processor, an advocate of a single model of learning into one who describes at least five distinctly different kinds, a proponent of the primacy of hierarchically organized intellectual skills into someone who now asserts that cognitive strageties serve as the executive processes which control intellectual skills, and these strategies may well be learnable only by discovery. The details and implications of this change in Gagné's position cannot be discussed adequately here, but they may well be profound. (It is likely that these changes which, to an American psychologist, seem comparable in magnitude to a religious conversion, will strike the European educator or psychologist as trivial indeed. Despite the shift in theoretical predilection, Gagné remains like most American educators, committed to the design of instruction even when discussing discovery.)

The remaining sections of this paper will describe several exemplars of information processing research on mathematical learning, thus exemplifying the manner in which such studies employ descriptions of behavior, systematic introspection and computer simulation to develop and test theories of mathematical learning and problem solving. That this approach has begun to dominate American research in this area will also be asserted and supported. We shall conclude with a proposal for research on teaching growing out of this same model, as augmented by other methods of studying thought and judgment, leaving to Professor Brophy the task of describing the manner in which the broad field of research on teaching has developed during the past ten years.

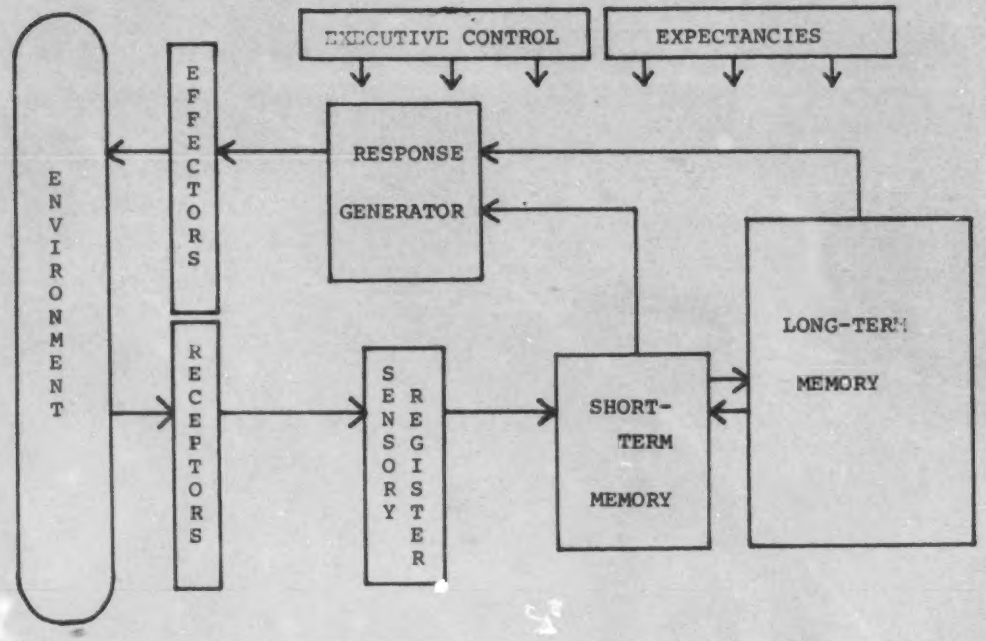
2. EXAMPLES OF INFORMATION-PROCESSING RESEARCH

What is an information-processing perspective on human learning and thinking? It emerges from a conception of cognition which treats the learner as a goal seeking

problem-solver whose ability to deal rationally with his environment is profoundly constrained by the intrinsic limitation of his capability for processing information. These limitations inhere in many stages of the cognitive process, including attention, selective perception, and most significantly, short-term memory and encoding for long-term memory storage and retrieval.

Thus, the "bounded rationality" (Simon, 1957) of the human information-processor results in his constructing a simplified model of real problem situations in order to cope with them. To understand why a person approaches a problem in a particular way, we must therefore understand his goals, the major characteristics of the task environment in which he works, and the transformation of that task environment into a cognitive problem space which reflects the limitations of his invariant information processing capacities.

The manner in which one approaches problems of mathematical learning from this perspective is reflected in the work of Greeno, to which we will devote extended attention. James Greeno (unpublished, 1976) has been studying the processes used by students proving triangles congruent. For the study, 24 protocols of successful proofs were collected from five students enrolled in a high school geometry class. The students were asked to talk their way through the proofs and respond to questions which the examiner might ask to help clarify his understanding of their thinking. Using the detailed descriptions collected from the student reports and the careful examinations of the problems and their possible solutions, some fundamental processes which might have been capable of generating the protocols are postulated. In this situation, the general strategy which was most apparent was that of searching for pairs of congruent parts until a recognizable pattern for proving triangles congruent became evident. This delay in setting a goal to use a particular theorem is not a familiar routine to the means-end analysis (Newell & Simon, 1972)



(From Gagne R.M. Essentials of Learning for Instruction. Dryden Press, 1974)

identified as the strategy in many of the tasks already studied. Once determined all of these details are given representation in a model which is expressed in a language appropriate to the problem.

The next stage of the investigation is the development of a computerized problem-solving system which is expected to simulate the performance of the students. To accomodate the strategies or processes extant in the models of student performances the computer program has been provided with three categories of processes:

(1) problem solving strategies similar to the means-end analysis mentioned earlier;

(2) a propositional inference capability which enables the system to use inferential statements to generate further information; and

(3) pattern recognition which tests for the presence or absence of particular features of the figures.

It is pattern recognition which enables the program to delay the choice of which theorem to use proving triangles congruent until sufficient information is avilable. Despite these capabilities the computer system does not possess the flexibility available to the human problem solvers, but it does realistically approximate their performances.

It is this computer program which constitutes a theory of geometry congruence proving, but it is the techniques of this study which illustrate the characteristics of an information processing approach to research. What are these characteristics? First, the procedure Greeno uses for collecting the information is often called process tracing (Shulman and Elstein, 1975). Second, the protocols were coded into a language and interpreted to represent the state of the problem and the processes used. This is all combined into a production system which is capable of gen-

erating the performance of the subjects. Third, a computer program is constructed to execute a production system which simulates the human problem solver. Fourth, a comparison is made between the protocols generated by the computer program and the protocols made by the human subjects to ascertain the adequacy of the theory.

Greeno's investigation is but one of many studies conducted under the rubric of problem solving research -- a field in need of a theory of mathematical problem solving. Such a theory must contain first, a description of how different kinds of problems are solved, and second, a taxonomic or logical analysis of the interrelations among problem types (Shulman and Elstein, 1975). Confronting this challenge, Greeno (in press) has suggested a typology -- not yet a taxonomy -- of problems. Although he acknowledges that many problems can not be classified as only one type, Greeno has, on the basis of the task demands, identified three types of problems along with the skills necessary to solve each type.

Description of Greeno's typology is inappropriate here, but the categories easily accommodate many problems of interest to the mathematics educator. Tasks which learners often face include concept attainment, rule or classification recognition, theorem proofs, and complex calculations. Greeno's typology supplies direction for further research.

In addition to identifying the process or strategies students use in a particular task environment, what are some of the values of this type research to mathematics educators? It may have similar implications for mathematics instruction that the medical inquiry studies (Shulman and Elstein, 1975) have had for instruction of medical students in the arts of diagnosis. Students adept at solving mathematical problems may be compared with physicians skilled in diagnosis. Knowing how "experts" behave has certain clear implications for how teachers assist students to perform

expertly. Medical students, for example, have traditionally been instructed to withhold forming hypotheses until all the evidence is collected. However, unlike the congruence provers, early hypothesis generation was found to be a universal characteristic of the sample of physicians and problems studied, and this awareness led to some changes in training medical students in diagnosis. Similar understanding of the processes of "expert" mathematical thinking may lead to new insights regarding mathematics teaching.

Another technique which falls within the scope of information processing research is directed at the isolation of a single cognitive process in mathematical learning. GROEN and PARKMAN (1972), GROEN and POLL (1973), and WOODS, RESNICK and GROEN (1975) have used response latencies -- differences in elapsed time -- to infer processes used in solving single-digit addition and subtraction problems. These are tasks where there is little behavioral or introspective evidence on which to produce protocols as in process-tracing studies such as Greeno's. A classical technique, chronometric analysis is based on the assumption that when a process can be broken down into identical steps, counting by ones in this case, reaction time will be a linear function of the number of counts needed to reach the answer. By collecting the latencies for a number of subjects solving a variety of problems and comparing the regression lines of possible processing models, the best fit suggests which model is most appropriate. Addition tends to be performed by setting the "counter" to the larger number and counting up, but subtraction requires a more heuristic approach -- counting up from the smaller or down from the larger depending on which is more efficient. Yet there are deviations from the regression lines suggesting alternative procedures for special cases such as recognizing doubles or the additive identity, and shifts in choices of processes suggesting a developmental trend.

Response latencies were also used in a study by ROSENTHAL and RESNICK (1974) to infer processes employed for correctly solving different classes of problems involving three numbers presented in verbal form but differing in order of mention, identity of unkown, and action verb.

There are several additional types of research on mathematical learning which reflect the information processing influence, but do not lead to computer simulation. The shift in approach to studying mathematics education is evident in the work of Lauren Resnick from the University of Pittsburgh Learning Research and Development Center. Having begun work on an elementary mathematics curriculum by building hierarchies of tasks, Resnick (Resnick and Glaser, 1975) is now applying information processing procedures and constructs to the study of problem-solving.

Arguing that intelligence is the ability to solve problems, Resnick, Glaser and collaborators have been studying what can be termed invention-problem-solving, focusing on the mechanisms operative "when in the absence of complete instruction, a person must "invent" a new solution to a problem by assembling previously learned skills". Resnick and Glaser have a generalized model (in flow chart format) to explain the behavior of problem-solving which includes three classes of processes:

(1) problem detection -- the awareness that a route to the solution is not available;

(2) feature scanning -- looking for cues and attempting to find a match between previously acquired knowledge and the task environment; and

(3) goal analyses -- redefining of goals and inventive routines to accomplish them.

To explore the behavior of the children who have been trained in the necessary skills but need to make transformations or inventions before their skills can be

applied, two problems were used: finding the area of a parallelogram and adding with multidigit numbers when carrying is required. A training procedure was derived from an analysis of the information processing steps required to build a simulation of the problem solving behavior. In general, the likelihood of invention was increased when strategies for planning ahead and considering alternative goals were incorporated.

Thusfar, we have seen that an information processing approach can employ introspective accounts of problem solving to generate a model of the cognitive steps employed by a subject, subsequently elaborating upon or testing the model through computer simulation. We have also observed how information-processing theories lead to particular kinds of task analyses, which break down the stages of information processing necessary to perform a task. These task analyses can then be tested experimentally using procedures such as measurement of response latencies or in instructional experiments employing as independent variables alternative conceptions of the strategies which ought to facilitate problem solving. SIMON (1975) has been particularly interested in the description and modeling of these strategies.

If strategies can be adequately represented, then instruction can be planned to enhance the mastery, workings, selection or retrieval of these strategies when needed by students. In this tradition, MAYER (1975) and colleagues conducted a series of studies to investigate which instructional variables influence the outcomes of learning. The research is directed by information-processing conceptions of that learning. The cohesiveness of these studies, quite rare in educational research, is brought about by continuing to use the same task which involves the learning of some probability concepts. To explore the consequences of changes in instructional variables, the investigators began with gathering empirical support

for one of three proposed models of internal processing. Indirect evidence was obtained by varying the instructional methods and examining the consequences in light of what each of the three models would predict. Attention was given the manner in which outcomes were measured, the use of the internal information processing variables, and the choice of instructional variables. One of the more important revelations, though not unexpected, was the indication of the need for mechanisms to receive material, to make available related previous experience to activate an assimilative set based on previous experience. Parallel efforts concerning the relation of prior experience in mathematics learning can also be seen in the work of LESH (1976) as he explores the meaning and use of Ausubel's advance organizers with college students learning finite geometries and groups, and with elementary school children learning motion geometry.

Finally, we cannot discuss the use of information processing approaches without reference to the exciting marriage of those methods and concepts with the concerns of Piagetian studies.

Within the Piagetian or developmental tradition it is worth commenting on the potential contribution of information processing in examining the transitions of cognitive development. SIMON (1972) advocates describing performance models of various tasks in order to trace their changes over time and determine the mechanisms which produce them. Examples of this modeling for class inclusion and weight seriation tasks can be found in KLAHR and WALLACE (1972) and BAYLOR and GASCON (1974). The expectation is that the research methods of information processing might bridge the gap between the hypothetical structures and processes of Piaget's theory and the levels of performance exhibited in the growing body of empirical studies.

Comparing a Piagetian logical model for class inclusion with an alternative information processing model, WILKINSON (1976) found support for the argument that inclusion errors result from the problem solving strategy employed rather than the logical deficits of a preoperational child. One strategy progosed for solving class inclusion problems is a counting procedure which forbids double-counting and seems to occur in children who have yet to become automated counters. Having postulated two strategies, the second allowing for more than one counting, Wilkinson ran a series of three experimental studies to determine whether the actual performances matched those predicted by logical constraints or the choice of processing strategy.

Concerned with children's causal reasoning but wishing to separate capability from possible learned responses, SIEGLER (1975) used an apparatus unfamiliar to the children to explore their causal reasoning powers. The outwart behavior suggested developmental differences so Siegler attempted to construct a model of the production system and then presenting different versions to the basic task to tease out which processes might account for the differences, whether misencoding, discerning relationships or varying standards for inference. Results were most interesting as the developmental differences seemed to be due to the ability to overcome the distracting influences. Developmental differences on tasks such as conservation, seriation, class inclusion or probability learning have had various explanations but Siegler appears to have a procedure which allows for testing competing theories by using what he labels interactional designs -- changing one aspect of the task at a time until the critical features are displayed.

A common thread emerging from all these studies is the centrality of an understanding of the <u>task</u> for understanding the intellectual processes needed to perform

the task. There is a tradition growing ever more powerful in psychology, traceable at least from Egon Brunswik, and including diverse contemporary scientists such as Herbert Simon, Peter Wason, and Robyn Dawes, which emphasizes the manner in which human problem solvers accomodate themselves to the demands of tasks.

3. RESEARCH ON TEACHING

It was observed earlier that among the serious flaws of the SHULMAN (1970) chapter, few were as serious as the ignoring of the contrality of teachers and teaching in the analysis of school learning. Were that chapter to be written today, it would not suffer from the same fault. The conception of teaching which I would propose also grows out of an information processing perspective. Rather than repeat what I have already written elsewhere, I call attention to the following appended materials for an elaboration of that perspective:

Shulman L.S. : Psychology of School Subjects; A Premature Obituary? Journal of Research in Science Teaching (see especially pp. 333-335), 1974

Shulman L.S.
Elstein A.S. : Studies of Problem Solving, Judgment and Decision Making. In F. Kerlinger (Ed.). Review of Research in Education III (see especially final section on implications for research on teaching), 1975

National Institute of Education, Teaching as Clinical Information Processing, Research planning panel, L.S. Shulman, Chairman.

FORSCHUNGEN ZUM PROZESS DES MATHEMATIKLERNENS

Gerhard Steiner

1. MATHEMATISCHE LERNPROZESSE AUS DER SICHT VON PIAGETS THEORIE

1.1 Zur Bedeutung einer neueren Genfer Arbeit (Inhelder et al., 1974)

In dem Maße als gewisse thematische Bereiche der modernen mathematischen Forschung (nicht zu verwechseln mit der "Modernen Mathematik" der Schule) mit entsprechenden Bereichen der Genfer genetisch-psychologischen Forschung übereinstimmen, hat man zuweilen bedenkenlos Piagets Entwicklungstheorie als die psychologische Grundlage für einen Mathematikunterricht postuliert. Daß Piagets Psychologie für einen schulischen Unterricht nutzbar gemacht werden kann, hat AEBLI (1951) gezeigt. Die "Didaktifizierbarkeit"von Piagets Theorie ist aber immer wieder angezweifelt oder bestritten worden, zuletzt von BUSSMANN (1974), der behauptet, daß weder explizit noch implizit die Möglichkeit gegeben sei, äußere Bedingungen für ein Auslösen und Steuern von Lernprozessen aus Piagets Theorie herzuleiten. Bussmann ist einer sehr orthodoxen Form der Genfer Theorie verpflichtet, die besagt, daß Lernprozesse, d.h. Assimilations- und Akkommodationsprozesse autonome, von außen unbeeinflußbare und sich selbst regulierende Prozesse seien, mit anderen Worten: daß sich die Dynamik der Entwicklung äußeren Anstößen entzieht. AEBLI (1968) hat in diesem Zusammenhang von einer Antinomie in Piagets Entwicklungstheorie gesprochen: Solange die geistige Entwicklung ein bestimmtes Niveau noch nicht erreicht hat, kann man (von außen, z.B. als Lehrer) nichts zur weiteren Entwicklung beitragen; ist das Niveau aber erreicht, braucht man nichts mehr beizutragen.

Für den Pädagogen und den Didaktiker scheint das Reifenlassen von Lernprozessen (oder umfassender: von Entwicklungsprozessen) die einzige Möglichkeit zu sein. Da Piagets entwicklungstheoretischer Ansatz aber eine Interaktionstheorie ist, der gemäß sich die Merkmale der geistigen Aktivität aus einer Interaktion des Individuums mit seiner Umwelt entwickeln, stellt sich die Frage, ob diese Interaktion nicht auch für Lernprozesse eine Schlüsselfunktion haben könnten. Tatsächlich haben sich viele Forscher nicht abhalten lassen, diese Frage zu bejahen und zu versuchen, Kinder mit bestimmten Umweltsituationen zu konfrontieren und dadurch kognitive Prozesse auszulösen. (vgl. dazu Beilin 1971; Montada 1970; Steiner 1973, 193-205). Alle diese Untersuchungen haben in jüngster Zeit durch eine Arbeit prominenter Genfer Forscher Unterstützung erhalten: INHELDER, SINCLAIR & BOVET (1974) zeigen, daß die ursprüngliche Quasi-Reifungstheorie der Genfer Schule von einer "Genfer Lerntheorie" abgelöst worden ist. Damit, so läßt sich gegenüber BUSSMANN (1974) argumentieren, werden alle zuvor schon aus Piagets Theorie abgeleiteten didaktischen Prinzipien gleichsam legitimiert. Wenn an diese Stelle die Arbeit von Inhelder et al. als eine wesentliche Studie betrachtet wird, so nicht aus unmittelbar unterrichtspsychologischen und didaktischen Gründen, sondern weil sie längst eröffnete Perspektiven bestätigt, die bis in die neueste Zeit in Frage gestellt worden sind. Inhelder et al. haben Lernversuche durchgeführt, in welchen sie die dynamischen Faktoren der Entwicklung durch Induzieren von Ungleichgewichten (Konflikte) beeinflußt haben, d.h. sie haben Folgen von Gleichgewichtungsprozessen ausgelöst. Piaget spricht in diesem Zusammenhang von "équilibration majorante", was man als "stets höher führende Aequilibration" interpretieren kann. Der langfristige

Entwicklungsprozeß, gleichsam die Makrogenese geistiger Strukturen, wird im Lernversuch mit Zeitraffereffekt als eine Mikrogenese der Prozesse des strukturellen Aufbaus beobachtbar. Die von den Genfer Forschern benützten Lehrverfahren sind die "exercices opératoires", die früher schon in experimentellen Untersuchungen angewandt worden sind (vgl. Montada 1968; Steiner 1974c). In der Didaktik wurde von "operatorischem Üben" (Aebli 1961) oder von operativer Gesamtbehandlung" (Fricke & Besuden 1970) gesprochen.

1.2 Piagets didaktisch fruchtbarster Begriff: la mise en relation

Der zentrale Ansatz für eine Didaktik der Mathematik (wie auch naturwissenschaftlicher Fächer) dürfte in Piagets Begriff der "mise en relation" liegen. "Mise en relation" bezeichnet das Verknüpfen von Verhaltens-(in der Genfer Psychologie meist Handlungs-)schemata oder anders: das Herstellen von Beziehungen und Zusammenhängen. Ohne auf die spezifische Problematik der Verinnerlichung (intériorisation) solcher Handlungskoordinationen einzugehen (vgl. dazu Steiner 1974a) muß erwähnt werden, daß bei diesen Koordinationen zweifellos auch sprachliche und Wahrnehmungsprozesse beteiligt sind, demnach nicht ausschließlich Handlungen. Die "mise en relation", der eine "abstraction à Partier de l'action" zugrundeliegt, steht im Gegensatz zur "lecture des données", die eine "abstraction à partir de l'objet" ist und folglich bloß Eigenschaften der Objekte "abstrahiert", was nicht zu einer Koordination von Handlungsschemata führt. Didaktisch lautet die Konsequenz: Im Unterricht müssen Situationen hergestellt werden (vom Schüler oder vom Lehrer), die eine "mise en relation" gestatten oder gar provozieren, die umgekehrt aber eine "lecture des données verhindern. STEINER (1973) analysiert unter diesem Gesichts-

punkt eine große Zahl von heute gängigen Mathematiklehrmitteln (wenn auch bei weitem nicht alle!) im deutschsprachigen Raum Europas und liefert damit eine psychologisch fundierte Inspektionsevaluation dieser Lehrgänge.

Im gleichen Zusammenhang verfolgt eine Arbeit von CANDREIA & STEINER (1972) eine andere Absicht: Es handelt sich um eine Sammlung von transkribierten Tonbandaufnahmen von Unterrichtsausschnitten (Anfangsunterricht in Mathematik mit Farbstäben), die unter psychologischen Gesichtspunkten für den praktizierenden Lehrer kommentiert werden. Die Arbeit erhebt erklärtermaßen keinen Anspruch auf Wissenschaftlichkeit (es handelt sich auch nicht um eine Untersuchung im streng wissenschaftlich-methodologischen Sinne), sondern stellt ein, wie man nachträglich sagen kann, geglücktes Verfahren zur Innovation des Mathematikunterrichts, d.h. der mathematisch-didaktischen Lehrerfortbildung dar. Der Wert der Arbeit liegt demnach in der Innovationsstrategie, die hier eingeschlagen worden ist. Der Bedarf nach ähnlichen Arbeiten, die eine Verbindung unmittelbarer Unterrichtssituation und Grundlagenwissenschaft herstellen, scheint mir auch für andere Bereiche des Mathematikunterrichts groß.

Mit einer Didaktik, die Piagets Begriff der "mise en relation" folgt, ist auch das Ziel einer Mobilitätsschulung im Denken des Kindes verbunden. Inhalte der sogenannten Modernen Mathematik werden erst in sehr bescheidenem Maße im Sinne einer "Genfer Lerntheorie" vom Kind erworben. Weit häufiger wird der Umgang mit Inhalten (z.B. aus der naiven Mengenlehre) auf der Basis einer empiristisch-behavioristischen Lerntheorie betrieben, deren didaktischer Ausdruck ein recht anspruchsloses wiederholendes Üben ist. Bislang liegen m. W. erst Pilotstudien über eine

operative Didaktik (im strengen Sinne) im Bereich moderner mathematischer Inhalte vor (z.B. eine laufende Untersuchung der Abteilung Pädagogische Psychologie der Universität Bern in 1. Klassen und die darauf aufbauende curriculare Planung für einen Umgang mit Baumdiagrammen). Ebenfalls als Pilotstudien desselben Instituts liegen Untersuchungen über das Induzieren und Lenken von mathematischen Lernprozessen bei Intelligenzschwachen (Hilfsschülern) und bei Rechenschwachen in Normalklassen vor. Entsprechend dem aus Piagets Theorie abgeleiteten Grundsatz für den Unterricht, die "mise en relation" zu fördern oder zu provozieren, die "lecture des données" aber zu vermeiden oder zuverhindern, lauteten die Arbeitshypothesen:

(1) Der schwache Schüler braucht nicht, wie oft angenommen wird, mehr Anschauung, sondern weniger, aber gezieltere.

(2) Das Verfahren des Abschirmens von wahrnehmbaren Gegebenheiten (Smedslund 1964; Bruner et al. 1971; vgl. auch Steiner 1973, 205-206 und 268-269) provoziert einen "mise en relation" in besonderem Maße.

(3) Ein begleitendes elementares Versprachlichen fördert die Verinnerlichung von Handlungskoordinationen.

(4) Die Mobilität des Denkens beim schwachen Schüler beginnt im Handlungsbereich; demnach sind Automatismen zu vermeiden.

Die Ergebnisse einer Therapie mit intelligenzschwachen Kindern auf der Basis dieser Hypothesen sind vielversprechend und sollten experimentell im Einzelversuch sowie im Klassenversuch wiederholt werden.

1.3 Lovells Untersuchungen auf der Sekundarstufe

Der Kongreßbericht des National Council of Teachers of Mathematics (Rosskopf, Steffe und Taback 1971) vereinigt eine Reihe von Untersuchungen zur Entwicklung des Verständnisses mathematischer Begriffe und Operationen aus der Sicht der Genfer Forschungen. Im Anschluß an PIAGET, GRIZE, SZEMINSKA & BANGs (1968) Ausführungen zur Epistémologie et psychologie de la fonction untersucht LOVELL (1971a) einige Aspekte der Entwicklung des Funktionsbegriffs. Das Verständnis dieses Begriffs gehört in den Bereich der Piagetschen formalen Operationen und der Elaboration von Operationen zweiter Ordnung (d.h. Operationen auf Aussagen anstatt konkreten Objekten). Lovell, der sich auf Untersuchungen von ORTON (1970) und THOMAS (1969) stützt, unterscheidet vier Entwicklungsniveaus im Aufbau des Funktionsbegriffs, wobei vor allem die Fähigkeit einer Verallgemeinerung des Begriffs hervortritt. Dabei zeigt es sich, daß schon Primarschüler gewisse Aspekte der Funktion (Abbildung) verstehen, wenn diese auf eine sehr konkrete Weise eingeführt werden. Ein volles Verständnis setzt allerdings ein entwickelteres und flexibleres formales Denken im Sinne Piagets voraus, also die Fähigkeit, Elemente nach dem Modell der Gruppenstruktur und der Kombinatorik zu verknüpfen (Piagets Gruppe INRC und die 16 binären Kombinationen; vgl. dazu Flavell 1963, 211-222; auch Steiner 1973, 136-141).

Im weiteren untersucht LOVELL (1971b), in welchem Maße Piagets strukturelles Modell als Erklärungsmodell beigezogen werden kann, um das Beweisverhalten, das Begründen mathematischer Zusammenhänge und das Erklären von mathematischen Prozessen bei Sekundarschülern zu interpretieren. Implizit stellt er fest, daß das strukturelle Modell von Piaget nicht ausreicht, um auftretende Schwierigkeiten etwa bei logischen Inklusionsproblemen zu erklären, obwohl grundsätzlich die Strukturen des konkret-operatorischen Niveaus der Entwicklung zur Meisterung der betref-

fenden Operationen ausreichen müßten. Lovell kommt in diesem Zusammenhang auf das Problem der Anschaulichkeit zu sprechen, die offenbar ein zusätzliches erklärendes Moment für die Interpretation von Schülerverhalten darstellt. Kognitionspsychologisch heißt dies, daß Piagets struktureller Aspekt nur eine Seite des Verhaltens erfaßt, daß die Problematik der Repräsentation der in einem Aufbauprozeß zu verknüpfenden Elemente aber auch - und zwar gesondert - mit berücksichtigt werden muß. (Letztlich führt diese Problematik zur Frage nach der psychischen Realität kognitiver Strukturen. Dazu Steiner 1974a.) Bruner hat in seiner Theorie der kognitiven Entwicklung und des kognitiven Lernens (1971, 1965) auf die Bedeutung der Repräsentationsmedien hingewiesen. "Es kommt nicht bloß auf die Verinnerlichung von Operationen an, sondern auch auf die immer ökonomischeren Repräsentationen der verschiedenen Gegebenheiten, an denen operiert wird" (Olson & Bruner 1974, 143).

2. MATHEMATISCHE LERNPROZESSE AUS DER SICHT DER NEUEREN KOGNITIONSPSYCHOLOGIE

2.1 Die "Berner Theorie"

Piagets Strukturalismus bezeichnet im wesentlichen die Tatsache, daß die Elemente des Verhaltens, auch der kognitiven Aktivitäten, in kohärenten Systemen organisiert sind. Nach Piaget können die konstituierenden Gesetzmäßigkeiten dieser Systeme formalisiert werden und zwar als "Gruppierungen" bzw. als Kleinsche Vierergruppen. Man kann heute aufgrund der neuesten kognitionspsychologischen Forschungsergebnisse die Ansicht vertreten, daß die Piagetschen Strukturen im Hinblick auf ihre Formalisierung eingeschränkte Sonderfälle von formal weniger eingeschränkten strukturellen Systemen sind. Die moderne Theorie der Repräsentation von Bedeutungen im Langzeitgedächtnis, d.h. die Theorie der Gedächtnisorganisation (Tulving & Donaldson 1972), spricht in diesem Zusammenhang von semantischen

Netzen. Im wesentlichen handelt es sich dabei um komplexe Verknüpfungen von Elementen (meist werden verbale Elemente ins Auge gefaßt) durch Relationen. Auf diese Weise lassen sich Bedeutungen von Begriffen oder Operationen darstellen. Durch eine zunehmende Dichte und Vielfalt von Verknüpfungen wird die laufende Erweiterung von Begriffen dargestellt, wie sie im Laufe der geistigen Entwicklung, aber auch im Verlaufe von schulischen Lernprozessen auftritt (man denke etwa an die Erweiterung des Zahlbegriffs von den natürlichen zu den ganzen, den rationalen und irrationalen und den imaginären Zahlen!). Innerhalb solcher Systeme oder Netzwerke lassen sich verschiedene Niveaus unterscheiden, auf denen ganze Netzteile zu neuen Elementen "verdichtet" werden, die ihrerseits wieder mit (neuen) Relationen verknüpft werden, was eine Verknüpfung höheren Niveaus darstellt. AEBLI (1976) bringt das Beispiel eines solchen Beziehungsgefüges und zwar aus dem Arithmetikunterricht der 1. Klasse (Abb. 1): Die Zahlen stellen die Elemente des Gefüges dar, die mathematischen Operationen die verknüpften Relationen.

In diesem Zusammenhang gehe ich noch einen Schritt weiter, um auf derselben theoretischen Grundlage darzustellen, wie dasselbe Problem auch mit Hilfe anderer Algorithmen gelöst werden kann (Abb. 2). Mobiliät im Denken eines Kindes bedeutet im Falle dieser arithmetischen Probleme das Verfügen über verschiedene Beziehungsgefüge zu ein und demselben Problem sowie über die Fähigkeit, sie zu durchschreiten. Die Darstellung zeigt, daß auch bei diesen einfachen Aufgaben Verknüpfungen bzw. Operationen höherer Ordnung auftreten.

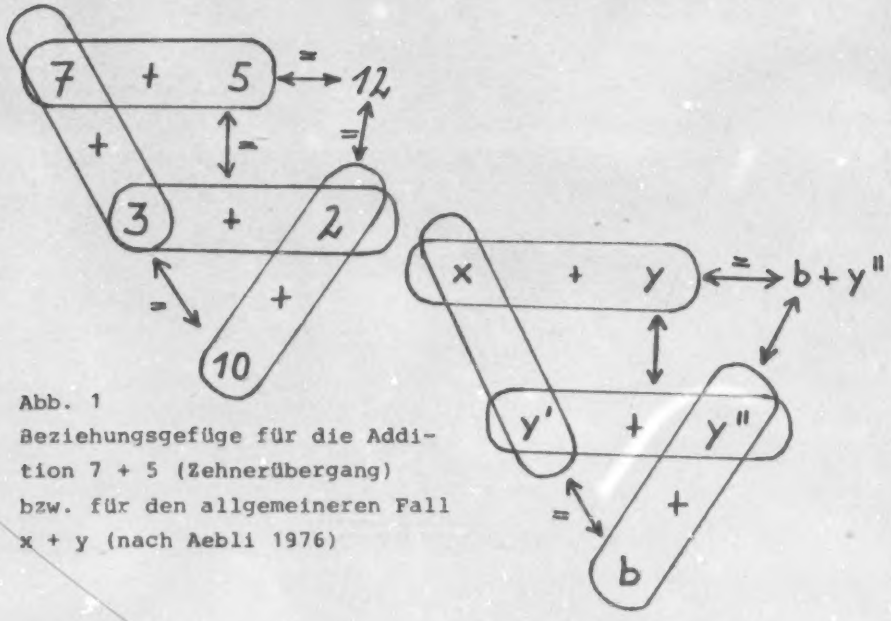


Abb. 1
Beziehungsgefüge für die Addition 7 + 5 (Zehnerübergang) bzw. für den allgemeineren Fall x + y (nach Aebli 1976)

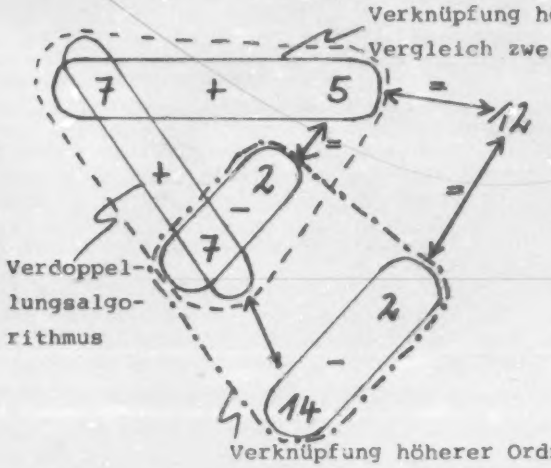


Abb. 2
Darstellung derselben Aufgabe wie in Abb. 1, Lösung mit Hilfe anderer Algorithmen: Verdoppelung und Analogie

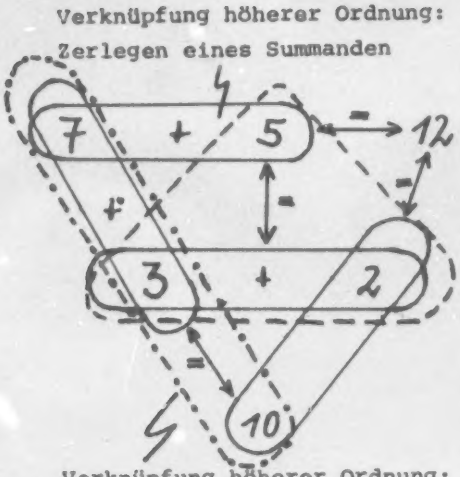


Abb. 3
Darstellung der höheren Verknüpfung in der in Abb. 1 dargestellten Aufgabe

2.2 Empirische Untersuchungen

Eine erste experimentelle Untersuchung, die auf diesem Modell der Gedächtnisorganisation beruht, ist diejenige von GREENEO (1972): Er geht von der Annahme aus, daß das Erlernen eines neuen Begriffs (in den von ihm geleiteten Untersuchungen ging es um die Binomialformel) eine Modifikation der bereits vorhandenen kognitiven Strukturen, d.h. eine Modifikation und Erweiterung des vorhandenen semantischen Netzes zur Folge hat. Demnach hängt der Lernprozeß von drei Faktoren ab:

(1) von den kognitiven Strukturen, über die der Lernende bereits verfügt,

(2) von den Eigenschaften des zu erlernenden Materials (hier der Formel) und

(3) von der Art und Weise, in der das neue Material dem

Lernenden präsentiert wird.

Ein erstes Ergebnis der Untersuchung besagt, daß die bereits vorhandenen Strukturen mit der Art der Darbietung interagieren: Versuchspersonen, denen viel Relevantes für das Erlernen des neuen Materials vertraut ist, mit anderen Worten, die über ein komplexes Netz von Elementen und Zusammenhängen aus der "Umgebung" des neu zu lernenden Stoffes besitzen, lernten besser mit einem entdeckenden Lernverfahren (d.h. einem entdecken-lassenden Lehren). Versuchspersonen dagegen, die wenig mit relevanten Informationen vertraut waren, lernten besser mit einem expositorischen Verfahren, in dem neue Begriffe ausführlich (explizit) "aufgebaut" wurden, d. h. in dem die neuen Elemente sorgfältig durch die relevanten Relationen miteinander verknüpft wurden (vgl. dazu auch Steiner 1976b). Im weiteren hat sich gezeigt, daß neu erworbene kognitive Strukturen auf zwei Diminsionen variieren, die typisch sind für das Netzmodell: einerseits spielen die Verknüpfungen der Elemente des neu zu lernenden Gegenstandes eine Rolle (internal connectedness), andererseits aber scheinen die äußeren Verknüpfungen (external connectedness), d.h. diejenigen des neu zu lernenden Stoffes mit den bereits bestehenden Netzteilen, eine bedeutende Rolle für das eigentliche Verstehen des neuen Inhalts zu spielen.

3. UNTERSUCHUNGEN ÜBER MIKROPROZESSE IM MATHEMATISCHEN LERNEN

Während die strukturellen Modelle zur Erklärung mathematischer Lernprozesse vorläufig noch kaum experimentell verifiziert sind, werden einzelne kognitive Teilprozesse in mathematischen Lern- und Problemlösesituationen experimentell untersucht. Dies gilt vor allem für visuelle Vorstellungsprozesse in mathematischen und geometrischen Aufgaben. Das Notieren mathematischer Zeichen (Zahlen, Operationszeichen usw.) kann als eine Art "verbaler Steno-

grafie" gelten (Skemp 1971). Vor allem fördern nach Skemp die Notationen ein visuelles Vorstellen im Problemlöseprozeß. HAYES (1973) hat die Bedeutung des notations-gebundenen Vorstellens in relativ einfachen mathematischen Problemen untersucht. Ihn hat vor allem interessiert, in welchem Maß Vorstellungen in die Struktur des Problemlöseprozesses eindringen. Eine seiner Untersuchungen hat unter Verwendung der Introspektion bei den Probanden eine erhebliche Vielfalt hinsichtlich Art und Ausmaß von visuellen Vorstellungen gezeigt. Dabei sind die äußeren Stimuli, d.h. die auf die Problemkarten geschriebenen oder gedruckten Zahlen oder Zeichen von besonderer Bedeutung: Die generierten Vorstellungen interagieren mit diesen physikalischen (äußeren) Reizen; diese werden teilweise vorstellungsmäßig modifiziert oder transformiert. Es entstehen Mischformen zwischen visuellen Wahrnehmungen und generierten Vorstellungen: "hybride Vorstellungsbilder". Die visuellen Vorstellungen scheinen nicht nur die Identitäten von Zahlen, sondern vor allem die räumlichen Eigenheiten und Relationen zwischen ihnen für den Problemlöseprozeß nutzbar zu machen. Die Identität der Zahlen könnte ebenso gut auditiv (durch auditive Vorstellungen bzw. durch sogenannt auditives rehearsal, d.h. Wiederholen) während eines Problemlöseprozesses in Bereitschaft gehalten werden. Viele Vorstellungen hängen, wie die Untersuchung vermuten läßt, mit früher eingeübten Rechenverfahren zusammen (etwa mit schriftlichen Algorithmen). Sie ersetzen diejenigen Aktivitäten, die ein Problemlöser ausführen würde, wenn er Bleistift und Papier zur Verfügung hätte. Vorstellungen lassen sich als "Erwartungen" von faktisch nicht eintreffenden oder nicht vorhandenen physikalischen Stimuli interpretieren. Dies ist auch NEISSERs (1976) Ansicht. Für eine solche Interpretation sprechen in Hayes' Untersuchungen gewisse Aussagen von Versuchspersonen, deren Vorstellungen/Erwartungen durch äußere Stimuli (z.B. durch Zahlen oder Zeichen, die an einer ungewohnten Stelle standen) gestört worden waren.

In längeren Problemlöseprozessen (von 3" Dauer und mehr) zerfallen die einmal generierten visuellen Vorstellungen wieder. (Die kognitionspsychologische Forschung gibt noch kein einheitliches Bild hinsichtlich der Zeiten, während der ein "Bild" bestehen bleibt. Posner et al., 1969, sprechen von sehr kurzen Zeiten: unter 2".) Hayes vertritt eine "Regenerierungsthese", d.h.: die für den Problemlöseprozeß benötigten visuellen Vorstellungen müssen immer wieder neu generiert werden. Die Ergebnisse eigener Untersuchungen (Steiner 1976a) mit geometrischen Aufgaben scheinen diese These zu bestätigen. Die Vorstellungen müssen nicht während des ganzen Problemlösungsprozesses "am Leben" erhalten bleiben, sondern werden regeneriert, wenn es die strukturelle Situation erfordert. Als Hauptergebnisse aus HAYES' (1973) Untersuchung lassen sich die folgenden zwei ansprechen:

(1) Eine Funktion des generierten Vorstellungsbildes ist es, räumliche Informationen zu tragen, die für die Problemsösung oder für Teile derselben nötig sind. Als solche kommen die Form von Zahlen und Zeichen, deren Anordnung und Orientierung sowie deren Größe in Frage.

(2) Ganz wesentlich - und didaktisch für die gesamte Problematik der Anschauung von zentraler Bedeutung (Steiner 1976a) - ist die Fähigkeit der Versuchsperson, "hybride Vorstellungsbilder" zu generieren, indem sie Teile der physikalischen Stimuluskomplexe (also der anschaulichen Gegebenheiten) mit Elementen der selber generierten Vorstellungen in ein einheitliches (vereinigtes) räumliches Schema bringt und dieses zur Problemlösung einsetzt.

In den oben erwähnten Untersuchungen (Steiner 1976a) sind visuelle Vorstellungen in kognitiven Aufbauprozessen untersucht worden. Es handelt sich dabei nicht um unmittelbar didaktisch relevante Versuche, sondern um einen Beitrag

zur kognitionspsychologischen Grundlagenforschung mit dem Ziel, mehr über das Wesen und die Funktion visueller Vorstellungsprozesse zu erfahren. Die Mikroprozesse (verbale und visuelle Teilprozesse wie Kodierungen im Kurzzeitgedächtnis, Umkodierungen, rehearsals usw.) scheinen bei geometrischen und arithmetischen Aufgaben recht komplex miteinander verknüpft zu sein. Eine klare Isolierung verbaler (oder verbal-auditiver) und visueller Prozesse ist bis jetzt noch nicht möglich gewesen. Vielmehr scheint das menschliche Informationsverarbeitungssystem recht flexibel zu sein, indem es nach Bedarf und äußeren Umständen die Kodierungsform (ob verbal-auditiv oder visuell) in einem gewissen Rahmen selber bestimmen kann. Das heißt z.B., daß eine Versuchsperson mehr verbalisiert, wenn es ihr aus experimentellen Gründen schwer gemacht wird zu visualisieren. Es sind allerdings noch erhebliche Anstrengungen zu unternehmen, bis hierüber sichere Aussagen gemacht werden können.

4. MATHEMATISCHES LERNEN AUS DER SICHT DER EINSTELLUNGSFORSCHUNG UND DER MOTIVATIONSPSYCHOLOGIE

AIKENs (1970) Sammelreferat zum Thema "Einstellungen gegenüber der Mathematik und dem Mathematiklernen" ist einerseits eine Fortsetzung von FEIERABENDs (1960) Zusammenfassung von Arbeiten zum gleichen Thema in den fünfziger Jahren und andererseits eine breit angelegte Antwort auf Fragen wie: Was hat die Forschung an Informationen über die Zusammenhänge zwischen Einstellungen von Schülern gegenüber der Mathematik und ihrem häuslichen Milieu, den Einstellungen ihrer Lehrer und den Inhalten der Curricula, die ihnen bereitgestellt werden, geliefert? Aikens Studie ist nicht bloß inhaltlich für den Pädagogen, den Didaktiker und den pädagogischen Sozialpsychologen interessant, sondern bietet auch dem forschenden Psychologen wertvolle methodologische Informationen hinsichtlich des Messens von Einstellungen (Beobachtungen und Interviews, Einstellungsskalen usw.). Aiken geht dabei auch auf die

Kritik an Einstellungsuntersuchungen ein, die sich vor allem auf eine exzessive Benützung korrelationsstatistischer Methoden und eine wenig sorgfältige Verwendung der Kovarianzanalyse bezieht. Inhaltlich stellt Aiken folgende Themen dar:

(1) Typische Verteilungen von bestimmten Einstellungen gegenüber der Mathematik auf verschiedene Schuljahre sowie ihre Stabilität im Verlaufe der Schulzeit.

(2) Das Verhältnis zwischen Einstellung und Leistung in Mathematik, wobei die Einstellung als eine Moderatorvariable betrachtet werden kann.

(3) Das Verhältnis zwischen Einstellungen, Persönlichkeitsvariablen und sozialen Faktoren. Dabei spielen vor allem Faktoren wie Angst, aber auch Eltern- und Mitschülereinfluß sowie geschlechtsspezifische Interessen eine Rolle.

(4) Zusammenhänge zwischen Persönlichkeitsmerkmalen, Ausbildungsstand und Einstellungen von Lehrern zu den Einstellungen und dem Verhalten von Schülern. Von besonderem Interesse sind diese Zusammenhänge im Hinblick auf Lehrerstudenten.

(5) Probleme, welche die Unterrichtsmethoden und die Curricula betreffen, wobei vor allem auch die Wirkung moderner Programme diskutiert wird.

(6) Die Frage, wie positive Einstellungen entwickelt und negative vermieden werden können.

Einige Aspekte von Aikens Aussagen sind zusammenfassend kommentiert in STEINER (1974b).

Im Zusammenhang mit Lehrerverhalten und Unterrichtsstil sowie mit anderen sozialen Faktoren, die sich auf Einstellungen zur Mathematik und zum Mathematiklernen auswirken und einen Einfluß auf den Lernerfolg in dieser Disziplin

haben, verdient ein neuerer Zweig der Motivationsforschung Erwähnung: die Attribuierungstheorie, die Theorie also, die sich mit der Ursachenzuschreibung im Falle von Erfolg oder Mißerfolg (u.a. in schulischen Bereichen wie dem Mathematikunterricht) befaßt. Dieser Forschungszweig, von HEIDER (1958) in Szene gesetzt, hat heute Entscheidendes für alle am Unterricht beteiligten über die Entwicklung der Leistungsmotivation zu sagen. In den neuesten Untersuchungen (Weiner 1974; Meyer 1973; Kun 1976), die allerdings nicht unmittelbar mit dem Mathematikunterricht zu tun haben, deren Ergebnisse aber für alle unterrichtlichen Bereiche bedeutsam sind, wurde vor allem herausgefunden, daß Ursachenzuschreibungen hinsichtlich Erfolg und Mißerfolg auf gewissen Voraussetzungen beruhen: Das Individuum (der Schüler) muß Kovariationsinformationen, d.h. Informationen über ein gleichsam paralleles Variieren von Ursachen und Wirkungen wahrnehmen und verarbeiten, muß kausale Schemata aufbauen und individuelle Differenzen wahrnehmen können. Die typischen Faktoren, auf denen eine Kausalattribuierung bei Erfolg oder Mißerfolg beruht, sind die persönliche Fähigkeit, die Anstrengung, die Aufgabenschwierigkeit und das Glück (bzw. der Zufall). Fähigkeit und Aufgabenschwierigkeit sind stabile Ursachen; Anstrengung und Glück (Zufall) sind unstabile. Man spricht von der Stabilitätsdimension. Fähigkeit und Anstrengung gehören inneren Kontrollbereichen an, Aufgabenschwierigkeit und Glück dagegen äußeren; die angesprochene Dimension wird als Kontrollbereich bezeichnet. Kontrollbereich und Stabilität sind die "kausalen Dimensionen", die sowohl die Erwartungen hinsichtlich Erfolg und die affektiven Reaktionen auf Erfolg bzw. Mißerfolg beeinflussen. Beide, Erwartungen und Affekte, sind Determinanten für die Motivationslage des jeweiligen Verhaltens - auch mathematischer Aktivitäten!

RELATED TO THE MATHEMATICS LEARNING PROCESS
RESEARCH 1970-1975

Marilyn N. Suydam

1. THE SETTING

In the United States, the years from 1970 through 1975 have been characterized by a continuing high level of interest in research on mathematics learning, despite a downgrading in priority for mathematics projects by federal agencies and corresponding lower levels of funding to support mathematics education research. Aiding and abetting the continuing interest is incrased communication among researchers. Groups of researchers have met not only to discuss the results of research -- as at the conference on Piagetian research at Columbia University -- but also to plan cooperatively the design of future research -- as in the postsessions conducted in 1973 and 1975 by the Special Interest Group for Research in Mathematics Education (affiliated with the American Educational Research Association), and in the workshops sponsored in 1975 (and 1976) by the University of Georgia's Center for the Study of Learning and Teaching Mathematics. The focus of the research discussed in such group meetings has been on the teaching and learning of mathematics, rather than on the methodology of research design or statistical analysis. The specific concerns have included logic, mathematical structure, computational algorithms, developmental psychology, problem solving, teaching strategies, measurement, geometry, and models for learning mathematics.

It is still true, as it has been for decades, that most of the research in the U.S. is done at the doctoral level. Very few persons can consider research as a full-time activity. Some researchers are involved in research only through planning studies with doctoral students. But the

number of researchers who also actively pursue their own research interests is increasing. Such involvement is important, for it permits long-range planning and the application of the experience and knowledge about the learner, the content, and research which accumulate from any continuing activity. Removal of the time restriction, which so often is concomitant with doctoral research, widens the type of study which can be formulated. Some researchers (e.g., Heimer at The Pennsylvania State University and Payne at The University of Michigan) involve many or all of their doctoral students in studies committed to an on-going purpose, increasing the probability that meaningful knowledge will accrue, rather than the random insights that occur from non-coordinated research efforts.

Interest in research is evidenced in directions other than from the mathematics education research community itself. One of the reasons the federal government established the ERIC (Educational Resources Information Center) system was to facilitate the collection of research reports. The ERIC Information Analysis Center at The Ohio State University (ERIC/SMEAC) added responsibility for mathematics education in 1970, and has facilitated not only the collection of documents for the ERIC files but has also published research materials including compilations of research, reports of research conferences, and discussions of research design. The limitations of ERIC are not always recognized, however: it is not the sole source of information on research. (For example, ERIC contains approximately 400 reports of research in mathematics education; my own files contain 6000 -- and are by no means complete.) Bibliographies built only from ERIC searches can be misleading.

The National Council of Teachers of Mathematics (NCTM) has evidenced increased interest in research during this period, with specific attention to bridging the gap be-

tween research and classroom practice. A number of research sessions are scheduled at each annual meeting and at most of the regional ("Name-of-Site") meetings. A Research Advisory Board has taken an active role in coordinating research activities within the Council and recently sponsored several publications, including one on Piagetion-oriented research and one on classroom applications from research on computational skills. And the Council began in 1970 to publish the Journal for Research in Mathematics Education, which serves as a much-needed outlet for the communication of research findings.

Another publication fills an important need in any research community. This is a journal of abstracts and critical reviews, Investigations in Mathematics Education. First sponsored by the School Mathematics Study Group (SMSG) and now by ERIC/SMEAC, it focuses attention on significance, design, and procedure questions, with the goal of promoting better research.

It is evident that the research being conducted in the U.S. is responsive to societal pressures of one form or another. This provides some responses from researchers to questions of immediate concern to teachers. However, it also tends to dissipate research energy into avenues where knowledge about continuing concerns does not necessarily accumulate. The balance between basic and applied research is tenuous, altered readily both by national educational priorities and by individual interests.

Despite continuing concern about their appropriateness, experimental paradigms are generally attempted, aimed at the collection of "hard" numerical data which can be subjected to computer analysis. Thus, use of such paradigms is abetted by the use of statistical techniques which are frequently more sophisticated than the design warrants. By nature, the experiment promotes the comparison of two or more treatments in a kind of contest whose focus is on determining

the "winner" (presumably to be awarded the title of "best"). This search for the "winner" is one of the less fruitful concomitants of applying experimental research paradigms. New statistical techniques have enabled a shift to study of the question "best under what conditions"?, but unfortunatily the potential of these techniques has been rarely observed.

Interest in the use of interviews as a means of adding depth to an experimental study and meaning to paper-and-pencil surveys has increased. Such words as "formative" and "exploratory" have been used with greater frequency; allied with this is interest in the "teaching experiment" used in the Soviet studies. In essence, researchers are trying to cope with the need to conduct studies which take into account the multitude of variables which constantly interact in the learning environment, while at the same time attain the goal of being able to generalize findings beyond the individuals with whom they are working.

While many curriculum developers have participated in international conferences or worked with non-American educators, comparatively few researchers in mathematics education have done so. Perhaps given impetus by such occurrences as the International Study of Achievement in Mathematics and by publication in the U.S. of the Soviet studies, researchers have realized that mathematics teaching may indeed differ, but that mathematics learning may involve similar patterns around the world. That all can profit from interaction and sharing what has been ascertained about how children learn mathematical ideas and how to teach mathematics more effectively -- as well as how to conduct research -- is obvious.

Thus, to summarize, the setting for research on mathematics education in the U.S. is promising: the need for coordination and cooperation is accepted and the interest is high. The outcomes of research, however, are in some

instances less satisfying.

2. THE OUTCOMES

The number of research studies reported per year from 1970 through 1975 in the annual listing in JRME is indicated in Table 1. Some of these studies are only tenuously connected with mathematics education; that is, mathematics is only one of several subject-matter fields considered (particularly at the elementary-school level). There is some overlap: some studies were first reported as dissertations and later as journal articles. But the listing is also limited since it does not contain research reported in books, monographs, ERIC documents, and of course, studies whose results have not been disseminated byond a local level (as well as studies which the compilers overlooked). In general, however, it would appear that approcimately 400 studies a year have been reported in the U.S. during the time period under consideration.

Table 1: Number of Research Studies 1970-1975 in Mathematics Education

	1970	1971	1972	1973	1974	1975
Summaries	14	10	13	9	9	7 ⁂
Articles	91	94	71	76	112	120 ⁂
Dissertations	223	322	335	299	264	300 ⁂

Important research outcomes are rarely indentified by single studies: significance is accumulated from a series of studies. In this section, research outcomes on ten topics which have been of much concern to resear-

⁂ Approximate data

chers are briefly summarized.[1] Each of these is important to understanding how to teach mathematics more effectively, yet the summary indicates that little more than the tip of the iceberg of knowledge is evident in most of these areas. From another viewpoint, however, the summary is positive: the parameters of the problem have been more clearly outlined in each case, and research is beginning to attack basic, underlying questions. The research process is clearly more complicated than it seems to be ... even though we thought it was going to be more complicated than it seemed (to paraphrase a comment made by Begle).

2.1 Organizing for instruction

Educators have long searched for the "perfect" organizational pattern to meet individual pupil needs and increase achievement. A large number of researchers have attempted to ascertain the efficacy or the superiority of various patterns, including departmentalization, team teaching, multigraded approaches, and self-contained classrooms. Attempting to isolate and measure the effects of any organizational pattern is extremely difficult, since factors such as content organization and especially teacher skill interact with the pattern. The definitions of the various patterns also tend to overlap -- what one person labels team teaching another defines as departmentalization, etc.

It is apparent from a review of the research that no general conclusion can be drawn regarding the relative efficiency of any one pattern for mathematics instruction. There appears to be no one pattern which will increase pupil achievement in mathematics; a proponent of any pattern can find studies that verify his stand. Achievement differences are affected more by other variables such as the skills of the teacher than by the organizational pattern.

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1) In some instances, statements were drawn from: Suydam, M.N.and Weaver, J.F., Using Research: A Key to Elementary School Mathematics. Columbus, Ohio: ERIC/SMEAC, 1975.

Perhaps the most important implication of the various studies is that good teachers are effective regardless of the nature of classroom organization.

2.2 Sequencing of instruction

Content organization and the sequencing of instruction have been a focus of many studies in the aftermath of the curriculum reform movement. Based on components of a learning theory by Gagne, hierarchies for various content have been developed and tested. There is some indication from such research that all subordinate skills are not retained after the superordinate skill is learned, but both the role and the ordering of subordinate skills is not clear. There is evidence from other studies that sequencing of mathematical content may not be as precise or important as was once commonly accepted. Evidence has also accumulated that learners tend to operate at various levels before they attain mastery. Yet a clear image of how concepts are formed or how learning is attained has not yet emerged from the research. While classical learning theories have been called into question (or discarded), newer theories have not filled the gap: they only provide some tentative bridges across it.

2.3 Teaching approaches

A conclusion similar to that for organizational patterns might be drawn from the research on teaching approaches. From the myriad studies on such facets as discovery vs. expository, mathematics laboratories, etc., we can conclude that there is no one approach that is more effective in effecting achievement than any other approach. Depending on such variables as age of learners, specific content, and especially skill of the teacher, any teaching approach can be effective. Little attention in most of these studies of teaching approach has been paid to the interaction of personality, background, skills, and

preference of both teacher and learner.[1] That one teacher may teach more effectively one way than another, just as one learner may learn more effectively one way than another, is just beginning to be a research interest. That there may be some content better taught, say, by an expository approach, and some by a mathematics laboratory approach, seems increasingly evident. What the teacher may need is not practice in one approach but experiences with many approaches, so that the most appropriate one for a teaching task may be selected.

Currently, there is much concern about individualizing instruction, with questions being raised about self-paces programs, in particular. From recent reviews it is apparent that the empirical evidence raises serious questions about the effectiveness of self-paces programs. Pacing is the easiest variable to manipulate in order to account for individual differences: it may not, however, be the most important one.

The topic of teacher effectiveness is a continuing thrust of research. For years, researchers explored the teacher's background, the number of mathematics courses the teacher had taken, the age of the teacher, the number of years of experience the teacher had, and a most of other factors, to determine correlates of teaching effectiveness. A few correlates were indeed found, such as warmth, enthusiasm, and managerial ability. In recent years -- at long last -- researchers have begun to look at the process of teaching, determining answers to questions such as, "What is it that

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1) While a number of studies have focused specifically on aptitude-treatment interactions, there have been few which have reached a finding other than "no significant differences". Many of these studies have been very brief, in an attempt to control extraneous variables. This brevity confounds the interpretation of the findings, but it may be that ATI research is based on a popular belief in how learners learn, and not on the realities of such factors as the learner's capability to adjust and assimilate.

goes on in a classroom?" and "What is the teacher doing which contributes significantly to the student's learning?" Information is accumulating on the amount of time actually spent on instruction (as opposed to managerial and other tasks, the answer is "little"), the kinds of questions actually asked (most are low level), and the quality of the interaction between teachers and learners.

2.4 Developmental stages (Piagetion-oriented)

A vast amount of research has been inspired by Piaget. However, the majority of these studies have been concerned with three questions:

- Are the stages proposed by Piaget valid? Most research has confirmed the existence of the stages: they do occur and they occur by and large in the order Piaget suggested. More variability in when the stages occur has been found.

- What is the effect of training? By the use of certain procedures, children can be trained to evidence a stage earlier than would be expected if they were untrained. However, there is no evidence that trained changes the overall developmental level of the children. The positive effect of training is rarely retained unless the child is already in transition from one level to another.

- Are Piaget's protocols accurate measures? In general, research indicates that unless the same questions and materials are used, results may differ.

Comparatively little research has explored the classroom implications of Piaget's theory. The topic on which the most research has been done is conservation. Conservation has been shown by several researchers to be related to work with counting, addition, and subtraction; several other researchers concluded that number conservation is

not a prerequisite to the development of mathematical understanding. The two may develop simultaneously.

Perhaps the most significant effect of Piagetian-inspired research is the shifting of concern from the curriculum back to the child. There is renewed recognition of developmental stages, of readiness, and of the importance of interviewing children to ascertain how they think.

2.5 Diagnosis and remediation

Concerns about diagnosis and remediation have accompanied interest in individualization and have been fostered by intervention programs for the disadvantaged. Some studies have focused on error analysis; there is a mass of information on errors pupils made on computation which could be a valuable resource for both researchers and curriculum developers. In other instances, programs have been developed and evaluated. It is possible to conclude that learners disadvantaged either academically or economically profit from such special attention. These learners differ in degree rather than kind of characteristics, with rate of learning just one variable to consider. They profit from physical involvement with materials (perhaps more than other learners do). Social relevance appears crucial, but the degree of mathematical meaning optimal for such learners is unclear.

A teach-test-reteach strategy is the only one consistently cited as effective.

2.6 Algorithmic learning

In terms of both the broad interpretation of learning how to learn and the restricted one of learning procedures for handling specific mathematical content, algorithmic learning has been a continuing concern. Some of the studies directed at specific computational algorithms have been among the most carefully designed pieces of research, al-

though they frequently suffer from the handicap of trying to determine which one algorithm is best, rather than under what conditions each of several might be best. Research on the broader meaning of algorithmic learning has frequently been characterized by obtuse reporting, so that it has been difficult for many other researchers to determine what the variables actually were and what is being done, much less what the importance of the finding is.

With the advent of hand-held calculators, the importance of knowing how learners process algorithms may be of even grater importance. Calculator algorithms as well as computational algorithms may need to be integrated.

2.7 Problem solving

Problem solving has attracted more attention from researchers than has any other topic in the mathematics curriculum. At the elementary-school level, verbal or word problems have been the center of attention, while at the secondary-school level heuristics have been increasingly explored. How to help children become better problem solvers has been answered with a variety of specific procedures that seem effective. We know with certainty only that it helps children to have some plan of attack, and that practice with many problems is needed. It has been much easier for researchers to determine what successful problem solvers have in common. Among the factors identified have been: ability to note likenesses, differences, and analogies; ability to visualize and interpret quantitative facts and relationships; skill in computation; ability to select correct procedures; reading comprehension skills; knowledge of fundamental mathematical concepts; a reflective personality; and a higher IQ (than poor problem solvers have!).

2.8 Use of materials

The role of manipulative materials in the learning of

mathematics is a topic which is of both theoretical and pragmatic concern. While many of the researchers have noted that the theories of Dienes and Bruner provided the instigation for the study, there is little solid testing of theory. It is increasingly evident that we need to analyze more carefully when materials should be used, with whom they should be used, what types should be used, and how they should be used.

- When? It may be that materials provide a foundation, but at some point they are no longer needed by children.

- With whom? It may be that alternative models should be available so that learners can select the ones most meaningful for them.

- What types? Another point of concern is whether the number of different materials -- or embodiments -- affects achievement. There is research evidence that one, two or three have essentially the same effect, but there is also evidence that those proficient in using more than one embodiment have a better understanding than those using fewer embodiments.

- How? Much evidence has accumulated which indicates that the use of manipulative materials appears to be essential in providing a firm foundation for developing mathematical ideas, concepts, and skills. In many of these studies, it is concluded that the learner must actively and individually handle the materials. There is, however, a growing body of evidence which indicates that perhaps it is as effective to have learners watch materials being manipulated (in a demonstration) as it is for the learners themselves to handle all materials.

In scores of studies, it appears that the learning of mathematics depends more on the teacher than on the materials used. Is this a "truth" -- or is it a function of the research pro-

cess? Currently, in a review of research sponsored by the National Institute of Education, an attempt is being made to answer such questions by analyzing and synthesizing outcomes across studies.

2.9 Achievement evaluation

With the movement toward accountability, and in reaction to criticisms of the curriculum reform movement, various extensive forms of achievement evaluation have been undertaken. The National Assessment of Educational Progress (NAEP) is the first attempt to obtain an unbiased sampling of the level of mathematical attainment in the U.S. Sophisticated item sampling procedures were used and stratified samples carefully selected. Unfortunately, few items could be administered on each bit of content, so there is some perhaprs unwarranted extrapolation from limited data in some reports of NAEP. At any rate, the purpose was to establish a baseline from which future comparisons could be made. Assessments in various states (notably Michigan) are contributin substantiating information that is of particular use to teachers within a given state. The Nacome-Report contains summaries of some of these efforts. The National Longitudinal Study of Mathematical Abilities (NLSMA) should also be cited; it resulted in the development of a wide variety of tests and sacles, as well as the collection of data whose usefulness may not yet be ended.

Using literature reviews, case-study approaches, and intensive data analysis, projects about to be funded will attend to patterns of change and effective ways of implementing change. One project will deal specifically with the dissemination and implementation of research and development efforts.

2.10 Attitudes and other non-cognitive factors

Attitudes and other non-cognitive factors are of secondary

concern in most studies, despite statements about the importance of the affective domain. From studies focused on non-cognitive factors, the following generalizations may be drawn:

- The majority of evidence indicates that relatively definite attitudes about mathematics have been developed by the time children reach the intermediate grades (approximately age 9).

- These attitudes tend to become increasingly less positiv as students progress through school. They are generally positive in the elementary school and appear to "peak" at approximately age 12.

- While we believe that attitude toward mathematics is related to achievement in mathematics, there is no consistent body of research evidence to support the popular belief that there is a meaningful or significant relationship between student attitude toward mathematics and student achievement in mathematics.

- Teacher are often viewed as being prime determiners of student attitude and achievement. There is some limited evidence that the effect of teachers' attitudes may be cumulative.

- Self-concept may be related to achievement in mathematics. Anxiety and various other factors may also play a role, but there is little definity in research.

- The interactive effect of learning style has little support in research, beyond the evidence from several studies that more reflective students tend to achieve better in mathematics.

It may be appropriate to conclude this section by citing ten generalizations formulated by Stevenson for Chapter 1 of the 37th NCTM Yearbook. He analyzed current research on young children's learning and cognitive development

as it may be related to the teaching of mathematics. It is apparent that many of these generalizations are appropriate for older learners as well as for young children.

(1) Wide individual differences exist in the abilities of children to learn and to solve problems, and these differences are complex and difficult to determine.

(2) Children may make errors because they attend to the irrelevant attributes of a situation.

(3) Young children are easily distracted by the presence of irrelevant information.

(4) Transfer is facilitated for the child if each rule has a wide variety of examples covering a wide range of extremes.

(5) Language may be helpful, but it is unnecessary for abstract thought.

(6) Children learn well through observation.

(7) Children use hypotheses, but their hypotheses may be inappropriate for the problem being presented.

(8) Difficult problems can be solved more readily if they are broken down into successively simpler components.

(9) Children may fail to solve problems because they cannot remember the components of the problem.

(10) The relationship between teacher and child is an important determinant of the child's performance.

LITERATUR / BIBLIOGRAPHY

Adda P.J. : L'incompréhension en mathématiques et les malentedus. Paru dans Educational Studies in Mathematics. Vol. 6, No. 3, 1975

Aebli H. : Didactique psychologique. Application à la didactique de la psychologie de Jean Piaget. Neuchâtel: Delachaux, 1951

Aebli H. : Grundformen des Lehrens. Stuttgart: Klett, 1961

Aebli H. : Über die geistige Entwicklung des Kindes. Stuttgart: Klett, 1963

Aebli H. : Grundformen des Lehrens (Neubearbeitung). Stuttgart: Klett (in Vorbereitung), 1976

Aiken L.R. : Attitudes toward mathematics Review of Educational Research 40: 551-596, Beilage, 1970

Aiken L.R. : Language factors in learning mathematics. Review of Educational Research 42: 359-386, 1972

Aiken L.R. : Ability and creativity in mathematics. Review of Educational Research 43: 405-432, 1973

Akamtsu T.
Thelen M. : A review of the literature on observer characteristics and imitation. Developmental Psychology 10: 38-47, 1974

Ames R. : Teachers' attributions of responsibility: Some unexpected non-definsive effects. Journal of Educational Psychology 67: 668-676, 1975

Ankori M.
Strauss S.
Orpaz N.
Stavy R. : Schooling effects on the development of proportional reasoning. Tel-Aviv University (unpublished), 1975

Armstrong J.R. : Representational modes as they interact with cognitive development and mathematical concept acquisition of the retarded to promote new mathematical learning. Journal for Research in Mathematical Education 3: 43-50, January 1972

Asblock
Herman : Current Research in Elementary School Mathematics. Lollier-Macmillan, 1970

Asbury C. : Selected factors influencing over- and under-achievement in young school-age children. Review of Educational Research 44: 409-428, 1974

Boersig T.M. : The effects of instruction in the enactive mode of representation on multivariable verbal problems encountered in elementary algebra (Purdue University, 1973) Dissertation Abstract International 34A: 3230, December 1973

Borg W.
Kelley M.
Langer P.
Gall M. : The mini course: A micro teaching approach to teacher education. London: Collier-Macmillan Limited, 1970

Brandt L.
Hayden M.
Brophy J. : Teachers' attitudes and ascription of causation. Journal of Educational Psychology 67: 677-682, 1975

Broomes D.R. : Psychological and Sociological Correlates of Mathematical Achievement and ability among Grade 9 students. Unpublished doctoral thesis. University of Toronto, 1971

Broomes D.R.
Schroeder T.L.
Payne R.D. : "Mathematics, Curriculum and Evaluation: Caribbean Experiences". Educational Development International 2: 67-73, April 1974

Broomes D.R.
Rahman J.
Richards C. : Mathematics Teaching Laboratories, Guyana: Ministry of Education and Social Development, August 1975

Brophy J. : Stability of teacher effectiveness. American Educational Research Journal 10: 245-252, 1973

Brophy J.
Good T. : Teacher-student relationships: Causes and Consequences. New York: Holt, Rinehart and Winston, 1974

Brophy J.
Evertson C. : Learning from teaching: A developmental perspective. Boston: Allyn and Bacon, 1976

Bruner J.S.
Kenney H.J. : Representation and mathematics learning. Monograph of the Society for the Research in Child Development 30: Nr. 1, 50-59, 1965

Bruner J.S.
Olver R.R.
Greenfield P.M.
et al. : Studien zur kognitiven Entwicklung. Stuttgart: Klett, 1972

Bruner J.S. : The Relevance of Education. Allen and Unwin, 1971

Bruner J.S. : A Study of Thinking. New York, 1956

Bussmann H. : Jean Piaget und die Mathematikdidaktik. Über die Möglichkeit, aus der Theorie Piagets Bedingungen für die Gestaltung des mathematischen Unterrichts abzuleiten. Zeitschrift für Pädagogik 20: 703-718, 1974

Barnes B. Clawson E. : Do advance organizers facilitate learning? Recommendations for further research based on an analysis of 32 studies. Review of Educational Research 45: 637-659, 1975

Bateson G. : Steps to an Ecology of Mind. Paladin Books, 1972

Baylor G.W. Gascon J. : An information processing theory of weight seriation in children. Cognitive Psychology 6: 1-40, 1974

Baylor G.W. Lemoyne G. : Foresight and Hindsight in an Information Processing Model of the Development of Seriation (University of Montreal).

Beckmann L. : Teachers' and observers' perception of causality for a child's performance. Journal of Educational Psychology 65: 198-204, 1973

Beilin H. : The training and acquisition of logical operations. In: Rosskopf M.F., Steffe L.P., Taback S., 81-124, 1971

Beilin H. : Future Research in Mathematics Education: The View from Developmental Psychology. Paper presented at Cognitive Psychology and the Mathematics Laboratory Symposium, Northwestern University, ERIC/SMEAC, February 1973

Beilin H. : Inducing conservation through training. In: Steiner G. (Hrsg., in Vorbereitung), 1976

Bishop A.J. : Research and the Teaching Theory Interface. Mathematics Teaching 60: 48-53, 1972

Bishop A.J. Whiterfield R.C. : Situations in Teaching. New York: McGraw-Hill, 1972

Blank M. : Teaching learning in the preschool: A dia - logue approach. Columbus, Ohio: Charles E. Merrill Publishing Co., 1973

Blank M. : Cognitive functions of language in the preschool years. Developmental Psychology 10: 229-245, 1974

Bledsoe J.C.
Purser J.D.
Frantz N.R. Jr. : Effects of manipulative activities on arithmetic achievement and retention. Psychological Reports 35: 247-252, August 1974

Bloom B.S.
Hastings J.T.
Madaus G.F. : Handbook on Formative and Summative Evaluation of Student Learning. New York: McGraw-Hill, 1971

Candreia A.
Steiner G. : Operatives Rechnen mit farbigen Stäbchen im 1. Schuljahr. Schule 72, Nr. 6-12; Schule 73, Nr. 1-4

Carmody L.M. : A theoretical and experimental investigation into the role of concrete and semi-concrete materials in the teaching of elementary school mathematics (The Ohio State University, 1970). Dissertation Abstract International 31A: 340-347, January 1971

Carver R. : The Coleman report: Using inappropriately designed achievement tests. American Educational Research Journal 12: 77-86, 1975

Case R. : Gearing the demands of instruction for the developmental capacities of the learner. Review of Educational Research 45: 59-87, 1975

Case R. : A Cognitive Model of a Classroom Task: The Missing Addent Problem (The University of California, Berkeley)

Chang S.
Raths J. : The school's contribution to the cumulating deficit. Journal of Educational Research 64: 272-276, 1971

Chapman E. : The Process of Learning Mathematics. Pergamon Press, 1972

Chapman R.H. : The development of children's understanding of proportion. Child Development 46: 141-148, 1975

Clark D. : Teaching concepts in the classroom: A set of teaching prescriptions derived from experimental research. Journal of Educational Psychology Monograph 62: No. 3, 1971

Clifford M.M. : Motivational effects of competition and goal-setting in reward and non-reward conditions. Journal of Experimental Education 39: 1971

Collins
Allan W.
Bolt
Beranek
Newman : An Information Processing Analysis of Tutorial Dialogues.

Collis K.F. : The Development of Formal Reasoning. University of Newcastle, NSW, Australia

Collis K.F. : Concrete and Formal Operations in School Mathematics. Australian Council for Educational Research, 1975

Copeland R.W. : How Children Learn Mathematics. Teaching implications of Piaget's Research. Collier-Macmillan, 1973

Craig G.L.
Myers J.L. : A developmental study of sequential two choice decision making. Child Development 34: 483-493, 1963

Cransac J.
Dauvisis M.C. : La rigueur du professeur de mathématique et la notation. Paru dans Bulletin APM No. 300, septembre 1975

Cunningham W. : The impact of student-teacher pairings on teacher effectiveness. American Educational Research Journal 12: 169-190, 1975

Curry R.D. : Arithmetic achievement as a function of concrete, semi-concrete, and abstract teaching methods (George Peabody College for teachers, 1970). Dissertation Abstract International 31A: 4032-4033, February 1971

Davies C. : Development of the probability concept in children. Child Development 36: 779-788, 1965

Denney D. : Modeling effects of conceptual style and cognitive tempo. Child Development 43: 105-120, 1972

DeVries D.
Edwards K. : Learning games and student teams: Their effects on classroom process. American Educational Research Journal 10: 307-318, 1973

Dodson J.W. : Characteristics of Successful Insightful Problem Solvers (University of Georgia, 1970). Dissertation Abstracts International 31A: 5928; Also NLSMA Report No. 31, May 1971

Domino G. : Interactive effects of achievement orientation and teaching style on academic achievement. Journal of Educational Psychology 62: 427-431, 1971

Douglas J. : Understanding Everyday Life. Rottgedge and Kegan P., 1973

Dowaliby F. Schumer H. : Teacher centered versus student centered mode of college classroom instruction as related to manifest anxiety. Journal of Educational Psychology 64: 125-132, 1973

Dunkin M. Biddle B. : The study of teaching. New York: Holt, Rinehart and Winston, 1974

Dusek J. : Do teachers bias children's learning? Review of Educational Research 45: 661-684, 1975

Duval R. : Rôle du langage dans l'apprentissage de la géométrie en quatrième. Tirage IREM de Strasbourg, 1972

Duval R. : Perspectives pour une pédagogie du raisonnement abstrait. Tirage IREM de Strasbourg, 1973

Dwyer C. : Influence of children's sex role standards on reading and arithmetic achievement. Journal of Educational Psychology 66: 811-816, 1974

Edwards K.J. DeVries D.L. Snyder J.P. : Games and teams. Simulation and Games 3: 247-269, 1972

Egan D.E. Greeno J.G. : Acquiring cognitive structure by discovery and rule learning. Journal of Educational Psychology 64: 85-97, 1973

Ehrenpreis W. Scandura J. : The algorithmic approach to curriculum construction: A field test in mathematics. Journal of Educational Psychology 66: 491-498, 1974

Engel A. : Teaching probability in intermediate grades, in Rade L. (Ed.). The teaching of probability and statistics. Stockholm: Almquist u. Wiksell, 87-150, 1970

Entwisle D. Webster M. : Raising children's performance expectations. Social Science Research 1: 147-158, 1972

Farnham-Diggory S. : Cognitive processes in education: A psychological preparation for teaching and

curriculum development. New York: Harper and Row, 1972

Feierabend R.L. : Review of research on psychological problems in mathematics education. In: Research problems in mathematics education. U.S. Office of Education, Cooperative Research Monograph No. 3, 3-46, 19[illegible]

Feldman L. : Mathematical Learning. New approaches to the teaching of young children. London: Gordon and Breach, 1969

Fennema E.H. : "The Relative Effectiveness of a Symbolic and a Concrete Model in Learning a Selected Mathematical Principle". Journal for Research in Mathematics Education 3: 233-238, November 1972

Fennema E.H. : Models and Mathematics. Arithmetic Teacher 19: 635-640, December 1972

Fiedler M. : Bidirectionality of influence in classroom interaction. Journal of Educational Psychology 67: 735-744, 1975

Firestone G.
Brophy N. : Longitudinal investigation of teacher-student interactions and their relationship to academic performance. Journal of Educational Psychology 67: 544-550, 1975

Fischbein E.
Pampu J.
Mânzat J. : L'intuition probabiliste chez l'enfant. Enfance 2: 193-208, 1967

Fischbein E.
Pampu J.
Mânzat J. : Comparison of ratios and the chance concept in children. Child Development 41: 377-389, 1970a

Fischbein E.
Pampu J.
Mânzat J. : Effects of age and instruction on combinatory ability in children. The British Journal of Educational Psychology 40: 261-270, 1970b

Fischbein E.
Bârbat J.
Mânzat J. : Intuitions primaires et intuitions secondaires dans l'initiation aux probabilités. Educational Studies in Mathematics 4: 264-280, 1971

Fischbein E. : The intuitive sources of probabilistic thinking in children. Dordrecht-Boston: D. Riedel Publishing Company, 1975

Fischbein E.
Mânzat J.
Bârbat J. : Dezvoltarea capacitâtii de investigare experimentalâ elexi. Revista de Psihologie 4: 409-424, 1975

Fischbein E. : The Intuitive Sources of Probabilistic Thinking in Children. Reidel, 1975

Flavell J. : The developmental psychology of Jean Piaget. New York: Van Nostrand, 1963

Gagné R.M. : Essentials of Learning for Instruction. Dryden, 1974

Gagné R.M.
Briggs L. : Principles of instructional design. New York: Holt, Rinehart and Winston, 1974

Gattegno C. : The Universe of Babies. New York: Educational Solutions, 1973

Gattegno C. : The Common Sense of Teaching Mathematics. New York: Educational Solutions, 1974

Glaeser G. : Réflexions sur le fiasco pédagogique et les conduites de sous-compréhension. Paru dans la Gazette des mathématiciens 2: 94, Juillet 1975

Glaymann M.
Varga T. : Les probabilités à l'école. CEDIC, Paris, 1973

Goldberg E. : Probability judgement, by preschool children: Task, condition and performance. Child development 37: 156-167, 1966

Good T.
Brophy J. : Behavioral expression of teacher attitudes. Journal of Educational Psychology 63: 617-624, 1972

Good T.
Sikes J.
Brophy J. : Effects of teacher sex and student sex on classroom interaction. Journal of Educational Psychology 65: 74-87, 1973

Good T.
Brophy J. : Changing teacher and student behavior: An empirical investigation. Journal of Educational Psychology 66: 390-405, 1974

Good T.
Biddle B.
Brophy J. : Teachers make a difference. New York: Holt, Rinehart and Winston, 1975

Good T.
Grouws D. : Teacher rapport: Some stability data. Journal of Educational Psychology 67: 179-182, 1975

Goutard M. : Mathematiques sur Mesure. Paris: Hachette, 1970

Gras R.
Chevrolet D. : Contribution à l'étude du comportement mathématique d'élèves de la classe de troisième. Paru dans Revue Francaise de pédagogie 32: 1975

Greene D.
Lepper M. : Effects of extrinsic rewards on children's subsequent intrinsic interests. Child Development 45: 1141-1145, 1974

Greeno J.G. : On the aquisition of a simple cognitive structure. In: Tulving E., Donaldson W. (Eds.), Organization of memory. New York: Academic Press, 353-377, 1972

Greeno J.G. : Information Processing Analysis of Studying from Text (will be University of Pittsburgh).

Greeno J.G. : Indefinite goals in well-structured problems. (unpublished paper).

Greeno J.G. : Natures of problem-solving abilities. In W.K. Estes (Ed.), Handbook of Learning and Cognitive Processes (in preparation or press?)

Groen G.J.
Parkman J.M. : A chronometric analysis of simple addition. Psychological Review 79: 329-343, 1972

Groen G.J.
Poll M. : Substraction and the solution of open sentence problems. Journal of Experimental Child Psychology 16: 292-302, 1973

Hall V.
Turner R.
Russell W. : Ability of children from poor subcultures and two grade levels to imitate and comprehend crucial aspects of English: A test of the different language explanation. Journal of Educational Psychology 64: 147-158, 1973

Hall B.
Turner R. : The validity of the "different language explanation" for poor scholastic performance by black students. Review of Educational Research 44: 69-82, 1974

Hamblin R.
Buckholdt D.
Ferritor D.
Kozloff N.
Blackwell L. : The humanization processes: A social, behavioral analysis of children's problems. New York: Wiley-Interscience, 1971

Harshmen H.W.
Wells D.W.
Payne J.N. : "Manipulative Materials and Arithmetic Achievement in Grade 1". Arithmetik Teacher 9: 188-192, April 1962

Hayes J.R. : On the function of visual imagery in elementary mathematics. In: Chase W.G. (Ed.) Visual information processing. New York: Academic Press 177-214, 1973

Heider F. : The psychology of interpersonal relations. New York: Wiley, 1958

Hoem A. : Kunnskap og makt. Universitetet i Oslo, Oslo, 1972

Hoemann H.W.
Ross B. : Children's understanding of probability concepts. Child Development, 221-236, 1972

Hosie T.
Gentile J.
Carroll J. : Pupil preferences and the Premack principle. American Educational Research Journal 11: 241-248, 1974

House P.A. : Learning environments, academic self-concepts, and achievement in mathematics. Journal for Research in Mathematics Education 6: 244-252, November 1975

Hughes D. : An experimental investigation of the effects of pupil responding and teacher reaction on pupil achievement. American Educational Research Journal 10: 21-38, 1973

Hunt D. : Person-environment interaction: A challenge found wanting before it was tried. Review of Educational Research 45: 209-229, 1975

Inhelder B.
Sinclair H.
Bovet M. : Apprentissage et structures de la connaissance. Paris: P.U.F., 1974

Investigation in Mathematics Education. Vol 5, Vol 6, Vol 7 No. 1, 2, 3, Centre for Science and Mathematics Education and ERIC. Science, Mathematics and Environmental Education Clearing hose. The Ohio State University, Columbus, Ohio 43210

Jamison D.
Suppes P.
Wells S. : The effectiveness of alternative instructional media: A survey. Review of Educational Research 44: 1-68, 1974

Jencks C.
et al. : Inequality: A reassessment of the effect of family and schooling in America. New York: Basic Books, 1972

Jensen A. : Interaction of level I and level II abilities with race and socio-economic status. Journal of Educational Psychology 66: 99-111, 1974

Johnson D.W.
Johnson R.T. : Instructional Goal Structure: Cooperative, Competitive, or Individualistic. Review of Educational Research 44: 213-240, Spring 1974

Journal for Research in Mathematics Education, Vol 1-6 inc. National Council of Teachers of Mathematics. Reston, Va. 22091

Kagan S.
Madsen M. : Experimental analyses of cooperation and competition of Anglo-American and Mexican children. Developmental Psychology 6: 49-59, 1972

Kahneman D.
Tversky A. : Subjective probability: a judgement of representativeness. Cognitive Psychology 3: 430-454, 1972

Keddie N. : Tinker, Tailor... The Myth of Cultural Deprivation. Penguin Education, p. 12, 1975

Keller H.R. : Children's acquisition and reversal behaviour in a probability learning situation as a function of programmed instruction, internal-external control and schedules of reinforcement. Journal of Experimental Child Psychology 11: 281-295, 1971

Kendler T. : An ontogeny of mediational deficiency. Child Development 43: 1-18, 1972

Kieren T.E. : Manipulative Activity in Mathematics Learning. Journal for Research in Mathematics Education 2: 228-234, May 1971

Kilpatrick J.
Wirzup I. : Soviet Studies in the Psychology of Learning and Teaching Mathematics. Stanford California: School Mathematics Study Group (A.C. Vroman), Chicago: University of Chicago Press (series of volumes)

Kimble G.A. : Required reading for the profession. Contemporary Psychology 8: 613-614, 1975

King I.L. : A Formative Development of an Elementary School Unit of Proof. Journal for Research in Mathematic Education 4: No. 1

Klahr D.
Wallace J.G. : Class inclusion processes. On S. Farnham-Diggory (Ed.), Information Processing in Children. New York: Academic Press, 1972

Klein S. : Student influence on teacher behavior. American Educational Research Journal 8: 403-421, 1971

Kleinfield J. : Effective teachers of Eskimo and Indian students. School Review 83: 301-344, February 1975

Kline M. : Why Johnny Can't Add: The Failure of the New Math. New York: St. Martin's Press, 1973

Kounin J.
Gump P. : Signal systems of lesson settings and the task related behavior of preschool children. Journal of Edcational Psychology 66: 554-562, 1974

Kounin J.
Doyle P. : Degree of continuity of a lesson's signal system and the task involvement of children. Journal of Educational Psychology 67: 159-164, 1975

Kun A. : Der Einfluß der Kausalattribution auf die Entwicklung der Leistungsmotivation. In: Steiner G. (Hrsg.), in Vorbereitung, 1976

Langford F.G. Jr. : Some Computational Strategies of Seventh Grade Pupils. ERIC: ED 069 496, October 1972

Lesh R.A. Jr. : An interpretation of advanced organizers (and others). Journal for Research in Mathematical Education 2: 69-74, 75-81, 82-86, 87-91, 1976

Lipe D.
Jung S. : Manipulating incentives to enhance school learning. Review of Educational Research 41: 249-280, 1971

Lovell K. : Some aspects of the growth of the concept of a function. In: Rosskopf M.F., Steffe L.P., Taback S. (1971), 12-33, 1971a

Lovell K. : The development of the concept of mathematical proof in abler pupils. In Rosskopf M.F., Steffe L.P., Taback S. (1971), 66-80, 1971b

Lovell : Intellectual Growth and Understanding Mathematics

Lucas J.F. : The Teaching of Heuristic Probelm-Solving Strategies in Elementary Calculus. Journal for Research in Mathematical Education 5: No. 1, 1974

Maclure S. : Styles of Curriculum Development. Conference held at Allerton Park, U.S.A., 1971, O.E.C.D., 1972

Malpas L.G. : The Approach to Mathematics. A. & C. Black, 1969

Martin R. : Student sex and behavior as determinants of the type and frequency of teacher-student contact. Journal of School Psychology 10: 339-347, 1972

Mayer R.E.
Greeno J.G. : Structural differences between learning outcomes produces by different instructional

methods. Journal of Educational Psychology 63: 165-173, 1972

Mayer R.E. : Acquisition processes and resilience under varying testing conditions of structurally different problem solving procedures. Journal of Educational Psychology 66: 644-656, 1974

Mayer R.E.
Stheil C.C.
Greeno J.G. : Aquisition of understanding and skill in relation to subject's preparation and meaningfulness of instruction. Journal of Educational Psychology 67: 331-350, 1975

Mayer R.E. : Information processing variables in learning to solve problems. Review of Educational Research 45: 525-541, 1975a

Mayer R.E. : Different problem-solving competencies established in learning computer programming with and without meaningful models. Journal of Educational Psychology 67: 725-734, 1975b

Meckes R.C. : A study to Ascertain the Instructional Index and Questioning Strategy of Mathematics Teachers in Grade 6, and to Determine Their Relationship to Professional Characteristics and Situational Factors (Southern Illinois University, 1971). Dissertation Abstracts International 32A: 4245-4246, February 1972

Mellin-Olsen S.
Skemp R.R. : Qualitative Differences in Mathematical Thinking. Report to the Nuffield Foundation and the NAVF, Norway, 1972

Mellin-Olsen S.
Rasmussen R. : Skolens vold. Pax, Oslo, 1975

Mellin-Olsen S. : Instrumentalism as a Pedagogical Concept. Research paper to be delivered to the ICME 3, Karlsruhe, 1976

Messik S.J.
Solley Ch.M. : Probability learning in children: some exploratory studies. Journal of Genetic Psychology 90: 23-32, 1957

Meyer W.-U. : Leistungsmotiv und Ursachenerklärung von Erfolg und Mißerfolg. Stuttgart: Klett, 1973

Miller R.L. : Individualized Instruction in Mathematics: A Review of Research. Mathematics Teacher 69: in press, May 1976

Montada L. : Über die Funktion der Mobilität in der geistigen Entwicklung. Stuttgart: Klett, 1968

Montada L. : Die Lernpsychologie Jean Piagets. Stuttgart: Klett, 1970

Mussen P. : Carmichael's manual of child psychology. Two volumes, third edition. New York: Wiley, 1970

National Council of Teachers of Mathematics; Piagetion Cognitive Development Research and Mathematics Education (NCTM, 1970)

Naylor F.
Gaudry E. : The Relationship of Adjustment, Anxiety, and Intelligence to Mathematics Performance. Journal of Educational Research 66: 413-417, May-June 1973

Neisser U. : Imagery and perception. In: Steiner G. (Hrsg.), in Vorbereitung, 1976

Newell A.
Simon H.A. : Human Problem Solving. Engelwood Cliffs, New Jersey: Prentice Hall, 1972

Newell A. : You can't play 20-question with nature and win: projective comments on the papers of this symposium. In Chase W.G. (Ed.), Visual Information Processing. New York: Academic Press, 1973

Noelting G.
Cloutier R.
Cardinal G. : Stades et mécanismes dans la developments de la notion de proportion chez l'enfant et l'adolescent. Université Lavel, Quebec (Unpublished), 1975

Nolen P. : Reading non-standard dialect materials: A study at grades two and four. Child Development 43: 1092-1097, 1972

Ojemann R.H.
Maxey E.J.
Snider B.C. : Effects of guided learning experiences in developing probability concepts at the fifth grade level. Perceptual and Motor Skills 21: 415-427, 1965

Ojemann R.H.
Maxey E.J.
Snider B.C. : Further study of guided learning experience in developing probability concepts in grade five. Perceptual and Motor Skills 23: 97-98, 1966

Olson D.R.
Bruner J.S. : Learning through experience and learning through media. In: Olson D.R. (Ed.). Media and symbols: The forms of Expression, communication, and education. The Seventy-Third Yearbook of the National Society for the Study of Education. Chicago: The University of Chicago Press, 125-150, 1974

Orton A. : A cross sectional study of the development of the mathematical concept of a function in

secondary school children of average and above average ability. Master's thesis, University of Leeds, 1970

Ortony A. : Language isn't for people: On applying theoretical linguistics to practical problems. Review of Educational Research 45: 485-504, 1975

Overview and Analysis of School Mathematics Grades K-12 (NACOME Report). Washington: Conference Board of the Mathematical Sciences, National Advisory Committee on Mathematical Education, 1975

Paris J.A. : The relation of a Personality Trait and Game Conditions to Participant Learning (Syracuse University, 1970). Dissertation Abstrct International 32A: 102, July 1971

Parker R.K.
Rieff M.L.
Sperr S.J. : Teaching multiple classification to young children. Child Development 42: 1779-1789, December 1971

Patterson G. : A basis for identifying stimuli which control behaviors in naturalistic settings. Child Development 45: 912-919, 1974

Payne J.N. : Mathematics Learning in Early Childhood. Thirty-seventh Yearbook of the National Council of Teachers of Mathematics. Reston, Virginia: The Council, 1975

Peel E.A. : The Nature of Adolescent Judgement. Staples Press, 1971

Piaget J.
Inhelder B. : La Genèse de l'idée de hasard chez l'enfant. Paris: PUF, 1951

Piaget J.
Grize J.-B.
Szeminska A.
Vinh Bang : Epistémologie et psychologie de la fonction. Paris: PUF, 1968

Pluvinage F. : Notions de varible. Paru dans Bulletin Inter-IREM, No. spécial, Avril 1974

Pluvinage F. : Sur la présentation des programmes scolaires. Paru dans Bulletin APM No. 300, 1975

Pluvinage F.
Duval R. : Une grille d'analyse et son application à deux manuels scolaires. Paru dans Bulletin APM, No. 301, Décembre 1975

Pluvinage F. : Taxonomie des automatismes (en préparation).

Posner M.I.
Boies S.J.
Eichelmann W.
Taylor R.L. : Retention of visual and name codes of singel letters. Journal of Experimental Psychology Monograph 79: 1-16, 1969

Psychology of Mathematics Education - papers from the Chelsea Seminars (Centre for Science Education, Chelsea College, London)

Quay L. : Negro dialect and Binet performance in severely disadvantaged black four-year olds. Child Development 43: 245-250, 1972

Quay L. : Language dialect, age, and intelligence test performance in disadvantaged black children. Child Development 45: 463-468, 1974

Reiss S.
Dyhdalo N. : Persistence, achievement, and open-space environments. Journal of Educational Psychology 67: 506-513, 1975

Resnick L.
Glaser R. : Problem solving and intelligence. In Resnick L.B. (Ed.). The Nature of Intelligence. Hillsdale, New Jersey: Laurence Erlbaum Associates, 1975

Richards P.N.
Bolton N. : Type of Mathematics Teaching, Mathematical Ability and Divergent Thinking in Junior School Children. British Journal of Educational Psychology 41, 1971

Rosenthal D.J.A.
Resnick L.B. : Children's solution processes on arithmetic word problems. Journal of Educational Psychology 6: 817-825, 1974

Rosskopf M.F.
Steffe L.P.
Taback S. : Piagetian Cognitive-Development Research and Mathematical Education (Proceedings of a conference conducted at Columbia University). Washington: National Council of Teachers of Mathematics, 1971

Schoen H.L. : Self-Paced Mathematics Instruction: How Effective Has It Been? Arithmetic Teacher 23: 90-96, February 1976

Schultz D.
Dangel T. : The effects of recitation on the retention of two personality types. American Educational Research Journal 9: 421-430, 1972

Shepherd J.G. : A Study of the Development of Operational Thought. Ph.D. thesis, University of Newcastle, New South Wales, 1973

Shulman L.S. : Psychology and mathematics education. In: Begle E.G. (Ed.), Mathematics Education, Sixty-ninth Yearbook of the National Society

for the Study of Education. University of Chicago Press, 1970

Shulman L.S. Elstein A.S. : Studies of problem solving, judgment, and decision making: Implications for educational research. In: Kerlinger F.N. (Ed.), Review of Research in Education, III. Itasca, Illinois: Peacock, 1975

Siegel S. Andrews J. : Magnitude of reinforcement and choice behaviour in children. Journal of Experimental Psychology 63: 337-341, 1962

Siegler R. Liebert R. : Effects of presenting relevant rules and complete feedback on the conservation of liquid quantity. Developmental Psychology 7: 133-138, 1972

Siegler R.S. : An Information Processing Approach to Children's Causal Reasoning. A paper presented at SRCD Convention, Denver Colorado, April 1975

Siegler R.S. : Utility of Interactional Strategies in the Study of Formal Operations Reasoning. A paper presented at SRCD Convention, Denver, Colorado, April 1975

Siegler R. Klahr D. : Information Processing Analysis of Instructional Readiness on Balance-Scale Problems (Carnegie-Mellon University).

Simon H.A. : On the development of the processor. In: Farnham-Diggery S. (Ed.), Information Processing in Children. New York: Academic Press, 1972

Simon H.A. : The functional equivalence of problem solving skills. Cognitive Psychology 7: 268-288, 1975

Skemp R.R. : The psychology of learning mathematics. Harmondworth: Penguin, 1971

Skemp R.R. : The Psychology of Learning Mathematics. Baltimore: Penguin Books, 1972

Smedslund J. : Concrete reasoning: a study of intellectual development. Monograph of the Society for Research in Child Development 93: Vol. 2, Nr. 2, 2-39, 1964

Soar R. : Teacher behavior related to pupil growth. International Review of Education 18: 508-526, 1972

Sowder L. : The Influence of Verbalisation of Discovered Numerical or Sorting-Task Generalsations on short-term Retention in Connection with the Hendrix Hypothesis. Journal for Research in Mathematical Education 5: No. 3, 1974

Steele J.M.
Walberg H.J.
House E.R. : Subject areas and cognitive Press. Journal of Educational Psychology 66: 363-366, June 1974

Stein A.
Smithells J. : Age and sex differences in children's sex role standards about achievement. Developmental Psychology 1: 252-259, 1969

Steiner G. : Mathematik als Denkerziehung. Stuttgart: Klett, 1973

Steiner G. : On the psychological reality of cognitive structures: a tentative synthesis of Piaget's and Bruner's theories. Child Development 45: 891-899, 1974a

Steiner G. : Kind und Mathematik - Neuere Beiträge der Psychologie. Zeitschrift für Pädagogik 20: Nr. 5, 677-702, 1974b

Steiner G. : Kognitionspsychologische Voraussetzungen zur intrinsischen Evaluation von modernem Mathematikunterricht in der Grundschule. Referat gehalten am 28. Kongreß der Deutschen GfPs in Saarbrücken 1972. In: Eckensberger L. (Hrsg.) Bericht über den 28. Kongreß der Deutschen Gesellschaft für Psychologie. Band 5: Psychologie und Erziehungswissenschaft. Göttingen: Hogrefe 43-58, 1974c

Steiner G. : Multimediale Mikroprozesse im kognitiven Aufbau. Bericht über das Projekt des Schweizerischen Nationalfonds zur Förderung der wissenschaftlichen Forschung 9290.73 (in Vorbereitung), 1976a

Steiner G. : Buchbesprechung von Neber H. (Hrsg.) Entdeckendes Lernen. Weinheim: Beltz 1973. In: Zeitschrift für Pädagogik, Heft 1, 1976b

Steiner G. : Die Psychologie des 20. Jahrhunderts. Band 7: Piaget und die Folgen. München & Zürich: Kindler (in Vorbereitung), 1976c

Steverson H.W.
Weir M.W. : Variables affecting children's Performance in a probability learning task. Journal of experimental Psychology 57: 403-412, 1959

Strickmeier H.B.Jr.: An Analysis of Verbal Teaching Behaviors in Seventh Grade Mathematics Classes Grouped by Ability (The University of Texas at Austin, 1970.) Dissertation Abstracts International 31A: 3428, January 1971

Strohner H. Nelson K.E. : The young child's development of sentence comprehension: influence of event probability nonverbal context, syntactic form and strategies. Child Development 45: 567-576, 1974

Taffel S. O'Leary K. Armel S. : Reasoning and praise: Their effects on academic behavior. Journal of Educational Psychology 66: 291-295, 1974

Thomas H.L. : An analysis of stages in the attainment of a concept of function. PhD dissertation, Teachers College, Columbia University, 1969

Travers R. : Second handbook of research on teaching. Chicago: Rand McNaily, 1970

Tulving E. Donaldson W. : Organization of memory. New York: Academic Press, 1972

Tversky A. Kahneman D. : Availability: A heuristic for judging frequency and probability. Cognitive Psychology 3: 207-232, 1973

Tversky A. Kahneman D. : Judgement under uncertainty: heuristics and biases. Science 185: 1124-1131, 1974

Vance J.H. Kieren T.E. : Mathematics Laboratories - More Than Fun? School Science and Mathematics 72: 617-623, October 1972

Veldman D. Brophy J. : Measuring teacher effects on pupil achievement. Journal of Educational Psychology 66: 319-324, 1974

Walker D. Schafferzick J. : Comparing curricula. Review of Educational Research 44: 83-111, 1974

Ward W. Barcher P. : Reading achievement and creativity as related to open classroom experience. Journal of Educational Psychology 67: 683-691, 1975

Weaver J.F. : Some Factors Associated with Pupils' Performance Levels on Simple Open Addition and Subtraction Sentences. Arithmetic Teacher 18: 513-519, November 1971 (Also: School Science and Mathematics 72: 679-691, November 1972, and Journal for Research in Mathematics Education 4: 45-56, January 1973).

Weaver J. : The Symmetric Property of the Equality Relation and Young Children's Ability to solve open Addition and Subtraction Sentences. Journal for Research in Mathematical Education 4: No. 1, 1973

Weiner B. : Attribution theory, achievement motivation, and the educational process. Review of Educational Research 42: 203-216, 1972

Weiner B. : Achievement motivation and attribution theory. Morristown N.J.: General Learning Press, 1974

White R. : Research into learning hierarchies. Review of Educational Research 43: 361-375, 1973

Wilden A. : Systems and Structure. Tavistock Publications, 1971

Wilkinson A. : Counting strategies and semantic analysis as applied to class inclusion. Cognitive Psychology 8; 64-85, 1976

Williams J.D. : Teaching Technique and Primary Mathematics. National Foundation for Educational Research, 1971

Willis S.
Brophy J. : Origins if teachers' attitudes towards young children. Journal of Educational Psychology 66: 520-529, 1974

Wilson J.W. : NLSMA Reports. Stanford, California: School Mathematics Study Group. Vroman A.C., volumes 1-33

Woods S.S.
Resnick L.B.
Groen G.J. : An experimental test of five process models for subtraction. Journal of Educational Psychology 1: 17-21, 1975

Woodson M. : Seven aspects of teaching concepts. Journal of Educational Psychology 66: 184-188, 1974

Wright R. : The affective and cognitive consequences of an open education elementary school. American Educational Research Journal 12: 449-464, 1975

Yost P.
Sieger A.E.
Andrews J.N. : Non-verbal probability judgement by young children. Child Development 33: 769-780, 1962

Young M. : Knowledge and Control. Collier Mac Millam, 1971

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